A Central Limit Theorem for Fleming-Viot Particle Systems
Application to the Adaptive Multilevel Splitting Algorithm

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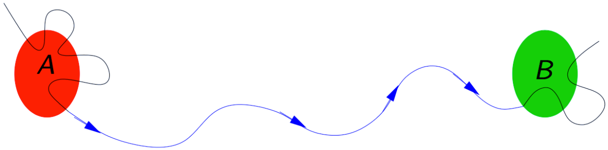
London 2018

Dynamical rare event problem

- $s \mapsto Y_s$ denotes a Markov continuous (diffusion) process in F; A and B denote two sets in F.



- Problem: Simulate a reactive trajectory, *i.e.* that starts from y_0 close to A and reaches B before going back to A.
- Problem: Rare event simulation $p \ll 1$.

Use of a Reaction Coordinate

- Reaction coordinate: one-dimensional continuous function

$$\xi : F \to \mathbb{R}$$

- Notation:

$$\{\xi > t\} := \{y \in F, \xi(y) > z\}$$
$$S_t := \inf(s \geqslant 0 | \xi(Y_s) = t)$$
$$S_A := \inf(s \geqslant 0 | Y_s \in A)$$

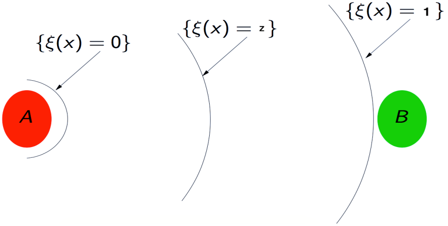
- For simplicity assume

$$A \simeq \{\xi < -1\} \ , \ B \simeq \{\xi > 1\}, \text{and}\, \xi(Y_0) = 0.$$

so that the considered rare event becomes

$$\{S_1 > S_A\} \ , \quad p = \mathbb{P}(S_1 > S_A)$$

Reaction Coordinate



- Define the 'score' of each trajectory by

$$\text{score} = \tau = \sup_{0 \leqslant s \leqslant S_A} \xi(Y_s)$$

- Multilevel/Importance Splitting: clone the trajectories with highest score, kill the other ones [Kahn and Harris (1951)].

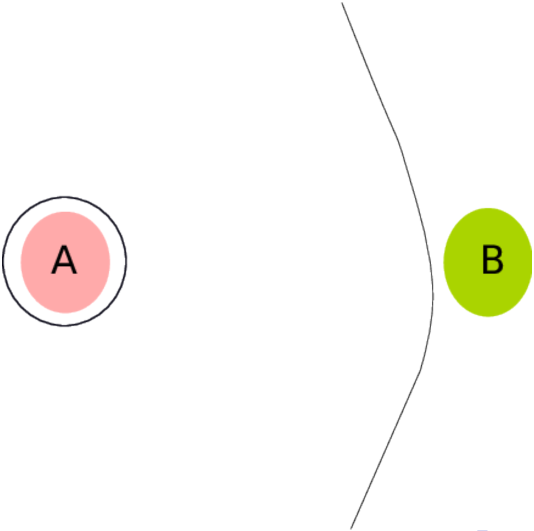
AMS algorithm

Definition (AMS algorithm)

At step $j \geqslant 0$ of the algorithm, we define a particle system $(Y^{1,j}, \cdots, Y^{N,j}) \in [C(\mathbb{R}_+, F)]^N$ defined by the following set of rules.

- *Initialization $j = 0$: consider N i.i.d. particles with law $(Y_s)_{0 \leqslant s \leqslant S_A}$.*
- *At step $j \geqslant 1$, kill the particle N_j with minimal score τ_j*
- *Splitting: the killed particle is given the state of particle M_j, uniformly picked among the $(N-1)$ remaining particles.*
- *The particle is then resampled starting from the entrance time in level τ_j: $S_{\tau_j}^{M_j, j-1}$.*
- *Set $j - 1 \to j$.*
- *And so on until all particles reach $\{\xi > 1\}$.*

Adaptive Multilevel Splitting



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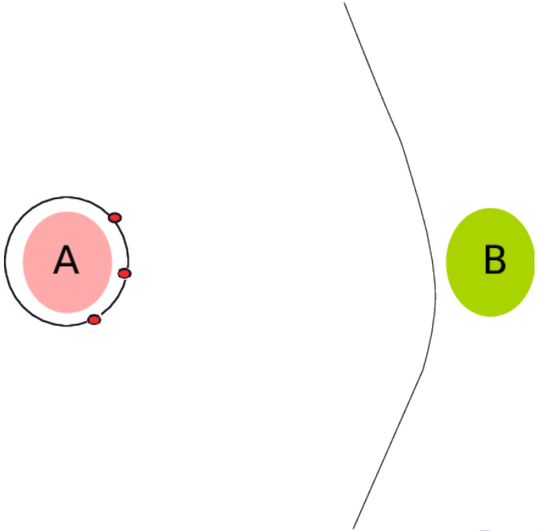

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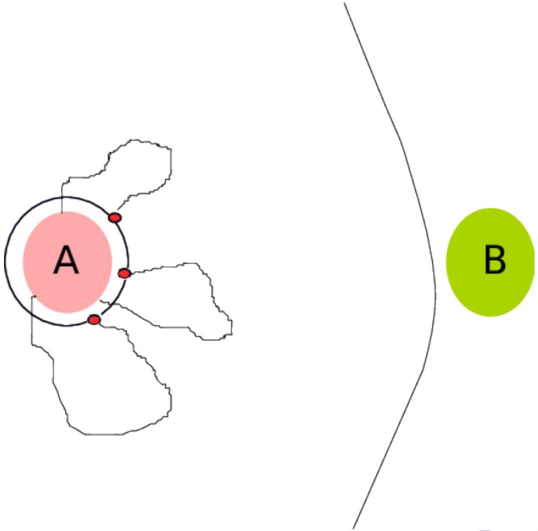

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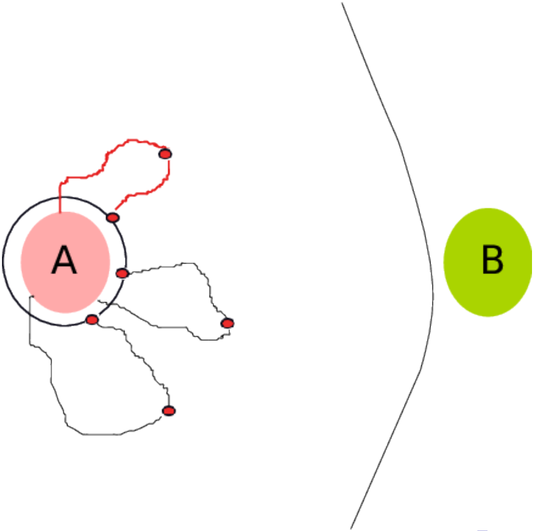

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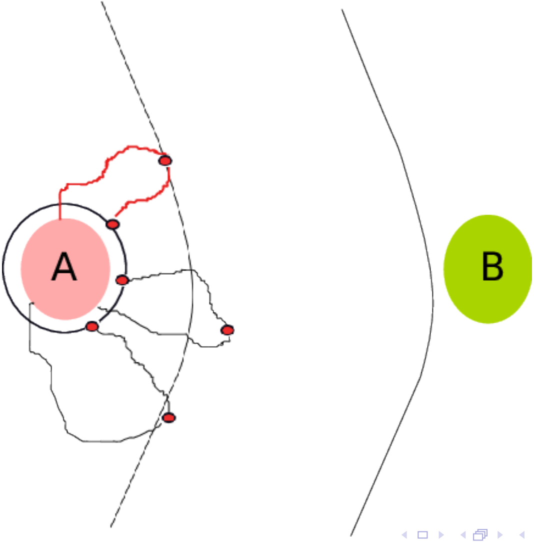

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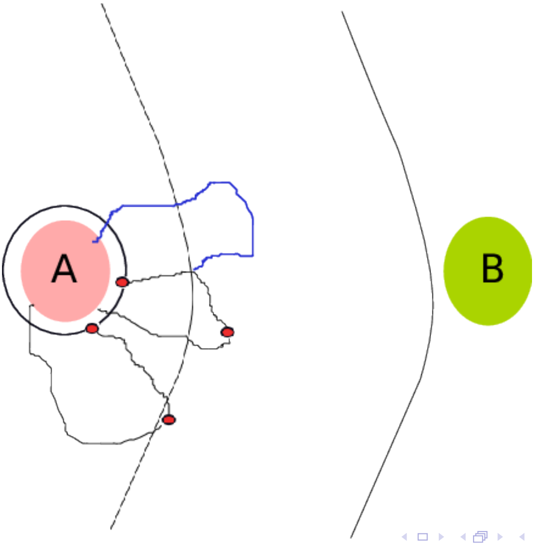

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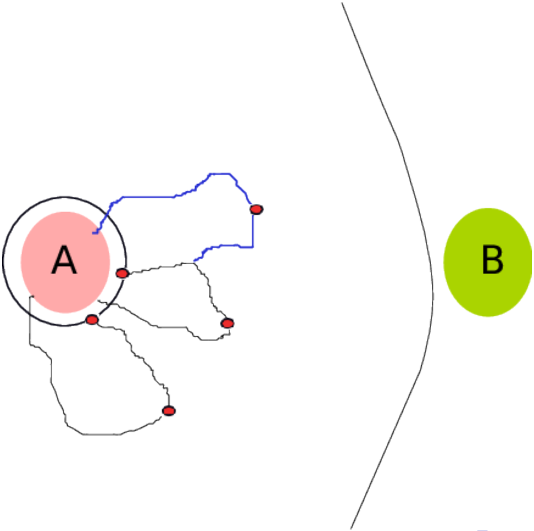

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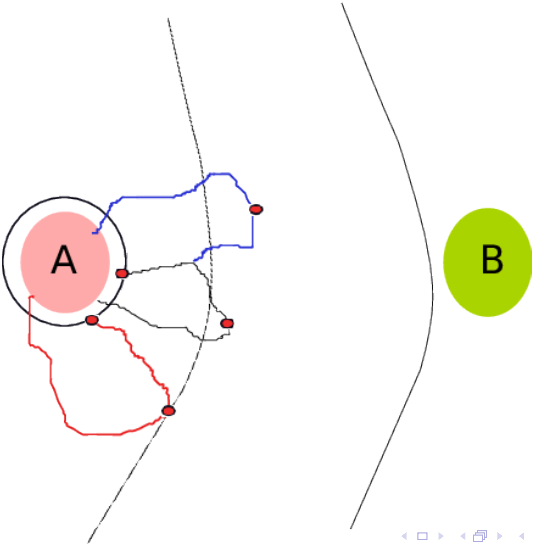


AMS yields Unbiased Estimator of Unormalized Quantity

- Denote by J_t the first iteration of branchings/steps so that all particles reach $\{\xi > t\}$.
- Denote empirical measure of particles paths at iteration J_t

$$\eta_t^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{Y_{s \geqslant 0}^{n, J_t}}$$

- Unbiased Estimates: For any pathwise observable (test function) φ, we have

$$\gamma_t(\varphi) := \mathbb{E}[\varphi(Y) 1_{S_t < S_A}] = \mathbb{E}[(1 - 1/N)^{J_t} \eta_t^N(\varphi)].$$

- Weak Law of Large Number ? Central Limit Theorem ?

The level-indexed process

- Key Idea: Assuming Y is strong Markov, define a new Markov process called the level-indexed process in $E := F \cup \{\partial\}$, the state space enhanced by a cemetary, by setting (NB: $\xi(Y_0) = 0$):

$$\begin{cases} X_t := Y_{S_t} & \text{if } S_t < S_A \\ X_t := \partial & \text{else} \end{cases}$$

- Can be extended to a time-homogenous process by setting

$$X_h := Y_{S_{\xi(x_0)+h}} \qquad \text{if } S_{\xi(x_0)+h} < S_A$$

- Can be extended to the pathwise case by setting $E := C(\mathbb{R}_+, F) \cup \{\partial\}$

$$X_t := Y_{0 \leqslant s \leqslant S_t} \qquad \text{if } S_t < S_A$$

The level-indexed process is a killed Process

Lemma

The level-indexed process $X = (X_t)_{t \geqslant 0}$ *is a Markov process in* $F \cup \{\partial\}$*, where the cemetary* $\partial \notin F$ *is absorbing (a trap). Moreover, the killing time* τ_∂ *satisfies*

$$\tau_\partial = \inf\{t \geqslant 0, X_t = \partial\} = \sup_{0 \leqslant s \leqslant S_A} \xi(Y_s) = score.$$

Moreover the conditional distribution on state space F *of interest becomes*

$$\eta_t = \mathcal{L}(X_t | \tau_\partial > t) = \mathcal{L}(Y | S_t < S_A),$$

and the probability of the rare event

$$p_t = \mathbb{P}(S_t < S_A) = \mathbb{P}(\tau_\partial > t).$$

Fleming-Viot Particle System

Definition (Fleming-Viot particle system)

The Fleming-Viot particle system $(X_t^1, \cdots, X_t^N)_{t\in[0,T]}$ *is the Markov process with state space* F^N *defined by the rules*

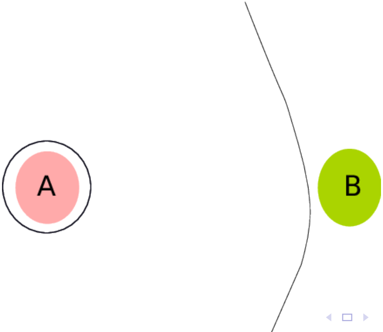
- *Initialization: consider N i.i.d. particles* $X_0^1, \ldots, X_0^N \overset{\text{i.i.d.}}{\sim} \eta_0$,
- *Evolution and killing: each particle evolves independently according to the law of the underlying Markov process X until one of them hits* ∂,
- *Splitting: the killed particle is taken from* ∂*, and is given instantaneously the state of one of the* $(N-1)$ *other particles (randomly uniformly chosen).*
- *and so on until final time T.*

Adaptive Multilevel Splitting

Lemma

The AMS algorithm mapped with the level-indexed process and stopped at iteration J_t (the first iteration when all particles reach $\{\xi > t\}$) is actually a Fleming-Viot particle system.

Proof Picture.



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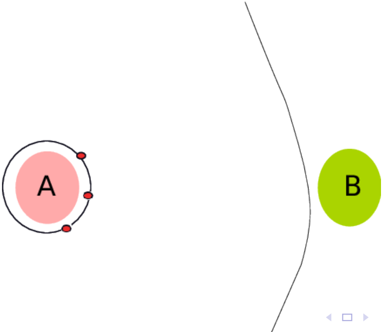

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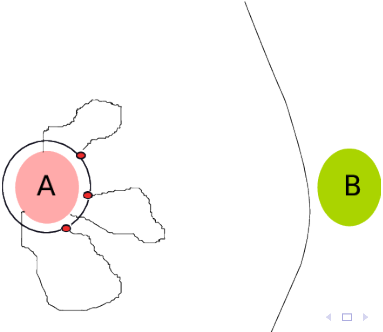

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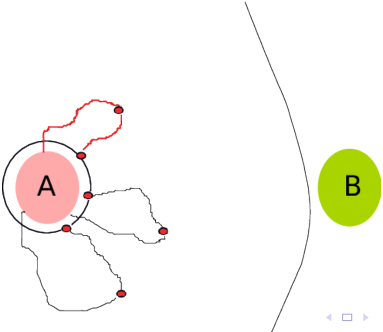

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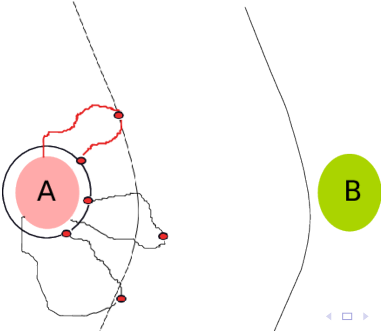

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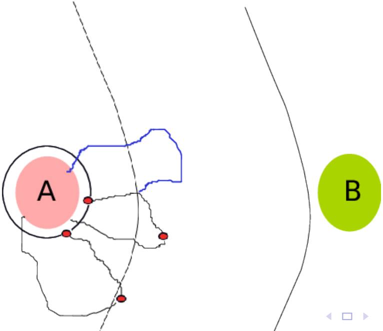

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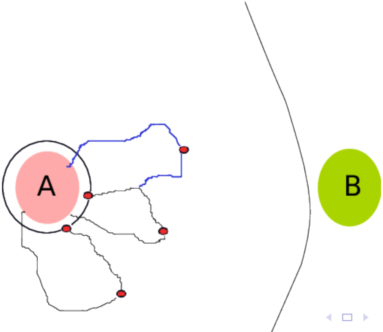

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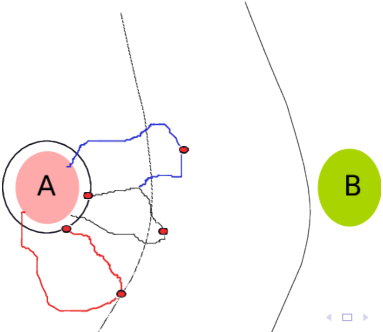

Some Important Estimators

Normalized distribution	proba of rare event	un-normalized distribution
$\eta_t = \mathcal{L}(X_t \mid X_t \neq \partial)$	$p_t = \mathbb{P}(X_t \neq \partial)$	$\gamma_t := p_t \eta_t$
Empirical	Empirical	Empirical
$\eta_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$,	$p_t^N = \left(1 - \frac{1}{N}\right)^{N \mathcal{N}_t}$,	$\gamma_t^N = p_t^N \eta_t^N$.

$\mathcal{N}_T := J_T / N$ is the average number of branchings per particle system.

Regularity Assumption

For $x \in F \cup \{\partial\}$, $0 \leqslant t \leqslant T$, consider the sub-Markovian semi-group

$$Q^t(\varphi)(x) := \mathbb{E}[\varphi(X_t)\mathbf{1}_{\tau_\partial > t} | X_0 = x].$$

Assumption (Non-synchronous jumps)

For any initial condition $X_{t_0} = x \in F$ and any $\varphi \in C_b(F)$:

(i) *the jump times of the càdlàg version of the martingale process $t \mapsto \mathbb{L}_t := Q^{T-t}(\varphi)(X_t)$ have an atomless distribution:*

$$\mathbb{P}(\mathbb{L}_{t^-} \neq \mathbb{L}_t | X_0 = x) = 0 \qquad \forall t \geqslant 0.$$

(ii) *The killing time has also an atomless distribution.*

Non-explosion Assumption

The 'non-synchronous jumps' Assumption is morally equivalent to: "martingale jumps" and branchings in the Fleming-Viot system are never simultaneous. In addition we ask:

Assumption (Non-explosion)

The Fleming-Viot system is non-explosive in the sense that the number of branching at any finite time is almost surely finite $\mathbb{P}(\mathcal{N}_T < +\infty) = 1$.

Example with Hard Obstacle (The originality of our result !)

$t \mapsto X_t$ process in E, and let F be an open domain with boundary $\partial F = \overline{F} \setminus F$. Let τ_∂ be the hitting time of $E \setminus \overline{F}$.

Proposition

(i) *The hitting time τ_∂ on boundary has an atomless distribution.*

(ii) *The process starting from boundary ∂F exits in open $E \setminus \overline{F}$ immediately almost surely.*

Then Assumption 'no synchronous jumps' is fullfilled.

Proposition (Grigorescu and Kang, 2012)

Assume that F is open and bounded in $\mathbb{R}^d$ with smooth boundary ∂F, and that X is a diffusion with smooth and uniformly elliptic coefficients. Then Assumption 'non explosion' holds true, as well as Assumption 'no synchronuous jumps'.

Central Limit Theorems

Theorem

Under Assumptions 'non-synchronous jumps' and 'non-explosion', for any $\varphi \in C_b(F)$ *one has the convergence in distribution*

$$\sqrt{N}\left(\gamma_T^N(\varphi) - \gamma_T(\varphi)\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_T^2(\varphi)),$$

where by definition

$$\begin{aligned}\sigma_T^2(\varphi) =& p_T^2 \mathbf{Var}_{\eta_T}(\varphi) - p_T^2 \ln(p_T)\, \eta_T(\varphi)^2 \\ & - 2\int_0^T \mathbf{Var}_{\eta_t}(Q^{T-t}(\varphi)) p_t dp_t .\end{aligned}$$

Central Limit Theorems

By Slutsky Lemma:

Corollary

Under Assumptions 'non-synchronous jumps' and 'non-explosion', for any $\varphi \in C_b(F)$, one has the convergence in distribution

$$\sqrt{N}\left(\eta_T^N(\varphi) - \eta_T(\varphi)\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_T^2(\varphi - \eta_T(\varphi))/p_T^2).$$

Besides,

$$\sqrt{N}\left(p_T^N - p_T\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \sigma_T^2(1_F) = -p_T^2 \ln(p_T) - 2\int_0^T \mathbf{Var}_{\eta_t}(Q^{T-t}(1_F))p_t dp_t.$$

Remarks on Asymptotic Variances

- The variance is the limit of the CLT of a discrete time Fleming-Viot particle system (CLT in Del Moral's book).

- Bounds on the probability estimator

$$-p_T^2 \log(p_T) \leqslant \sigma^2 \leqslant 2p_T(1-p_T) + p_T^2 \log(p_T)$$

- Lower bound: **(soft) killing time independent of the state**.
- Upper bound: **killing time mainly dependent on the initial condition**.
  Dominant term $2p_T(1-p_T)$ is twice the naive Monte Carlo variance.

Stochastic calculus with jumps

- Recall that one can integrate with respect ot 'semi-martingales' X = montonous processes + martingales as follows:

$$\int Y_{t-} dX_t \simeq \int Y_{t-}(X_{t+dt} - X_t)$$

- We then have the chain rule

$$d(X_t Y_t) = Y_{t-} dX_t + X_{t-} dY_t + d[X, Y]_t$$

where $t \mapsto [X, X]_t$ is an increasing process, bilinear with respect to vector space structure on X called the quadratic variation. Broadly speaking

$$[X, X]_t = \overset{\mathbb{P}}{\lim_{|t_{i+1} - t_i| \mapsto 0}} \sum_i (X_{t_{i+1}} - X_{t_i})^2$$

- If X is monotonous, $[X, X]_t$ is the sum of the squares of the jumps.

Stochastic calculus with jumps

- If $t \mapsto M_t$ is a (local) martingale, then $t \mapsto M_t^2 - [M, M]_t$ is again a local martingale.
- In the presence of jumps there ar plenty of 'quadratic variation'-like increasing processes $t \mapsto i(M)_t$ such that $t \mapsto M_t^2 - i(M)_t$ is a local martingale. For instance there is a unique $i(M)_t =< M, M >_t$ which is predictable.
- Example: let $t \mapsto M_t \in \{-1, 1\}$ be Poisson-like Markov + Martingale random walk process jumping up or down with proba $1/2$ at indep. expo. times. Then

$$[M, M]_t = \sum_{jumps} \mathbf{Var}(jump) = 1$$

$$< M, M >_t = dt$$

CLT for martingales with jumps

Theorem (Martinagle CLT (Ethier-Kurtz))

On a filtered probability space, let $t \mapsto m_t^N$ denote a sequence of càdlàg local martingales indexed by $N \geqslant 1$. Assume moreover that

(i) $m_0^N \xrightarrow[N\to+\infty]{\mathcal{D}} \mu_0$, *where μ_0 is a given probability on $\mathbb{R}$.*

(ii) *Vanishing jumps: One has* $\lim_{N\to+\infty} \mathbb{E}[\sup_{t\in[0,T]} |m_t^N - m_{t^-}^N|^2] = 0.$

(iii) *For each N, there exists an increasing càdlàg process $t \mapsto i_t^N$ such that*

$$t \mapsto \left(m_t^N - m_0^N\right)^2 - i_t^N$$

is a local martingale.

(iv) *Vanishing jump: The process $t \mapsto i_t^N$ satisfies* $\lim_{N\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} i_t^N - i_{t^-}^N\right] = 0.$

CLT for martingales with jumps

Theorem (Martinagle CLT (Ethier-Kurtz))

For the increasing càdlàg process $t \mapsto i_t^N$ (with vanishing jumps) such that

$$t \mapsto \left(m_t^N - m_0^N\right)^2 - i_t^N$$

is a local martingale:

(v) *!! Main Assumption !!:* There is a cont. and *incr. det. function $t \mapsto i_t$ s. t., $\forall t \in [0, T]$, $i_t^N \xrightarrow[N \to +\infty]{\mathbb{P}} i_t$.*

Then $(m_t^N)_{t \in [0,T]}$ converges in law (under the Skorokhod topology) to $(M_t)_{t \in [0,T]}$, where $M_0 \sim \mu_0$ and $(M_t - M_0)_{t \in [0,T]}$ is a Gaussian martingale, independent of M_0, with independent increments and variance function i_t (time changed Brownian motion).

CLT for martingales with jumps

In short, we need to construct martingales of order $1/\sqrt{N}$ from the particle system and ensure the convergence of 'a' quadratic variation of those martingales of order $1/N$.

Overview of the proof

- Key object: the càdlàg martingale

$$t \mapsto \gamma_t^N(Q) := \gamma_t^N\left(Q^{T-t}(\varphi)\right).$$

- Initial condition treated separately (easy).
- We will handle the distribution of $\gamma_T^N(Q)$ in the limit $N \to \infty$ by using a Central Limit Theorem for continuous time martingales.
- Not straightforward: the convergence of the quadratic variation $N[\gamma^N(Q), \gamma^N(Q)]_t$ is difficult (lots of IPPs !!).

Martingale decomposition [Villemonais 2014]

- The key martingale decomposition is the following:

$$\gamma_t^N(Q) = \gamma_0^N(Q) + \frac{1}{\sqrt{N}} \int_0^t p_{u^-}^N (d\mathbb{M}_u + d\mathcal{M}_u).$$

- With $\mathbb{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathbb{M}_t^n$ and $\mathcal{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathcal{M}_t^n$.
- $\mathbb{M}_t^n$ is the martingale contribution except for branching times of particle n.
- $\mathcal{M}_t^n$ is the martingale contribution at branchings only of particle n.
- No ambiguity, natural way to do this.

Orthogonality

The $2N$ martingales $\{\mathbb{M}_t^n, \mathcal{M}_t^m\}_{1 \leqslant n,m \leqslant N}$ are mutually orthogonal.

More specifically

(i) $[\mathbb{M}, \mathcal{M}]_t$ is a local martingale,

(ii)

$$[\mathcal{M}, \mathcal{M}]_t = \frac{1}{N} \sum_{n=1}^{N} [\mathcal{M}^n, \mathcal{M}^n]_t,$$

(iii) Moreover, if we note the 'intermediate' quadratic variation

$$(\mathbb{M}, \mathbb{M})_t = \frac{1}{N} \sum_{n=1}^{N} [\mathbb{M}^n, \mathbb{M}^n]_t,$$

then the process $[\mathbb{M}, \mathbb{M}]_t - (\mathbb{M}, \mathbb{M})_t$ is also a local martingale.

Ingredient (i): A 'key formula'

Lemma

The quadratic variation of martingales associated with the particles dynamics outside branchings can be related to

$$\gamma_t^N(Q^2) = \gamma_t^N([Q^{T-t}(\varphi)]^2)$$

through the key formula

$$p_{t-}^N \, d(\mathbb{M}, \mathbb{M})_t = d\gamma_t^N(Q^2) + Martingale$$

Ingredient (ii): L^2 apriori estimates

Proposition (Villemonais 2014, CDGR 2017)

For any $\varphi \in \mathcal{D}$, we have

$$\mathbb{E}\left[\left(\gamma_T^N(\varphi) - \gamma_T(\varphi)\right)^2\right] \leqslant \frac{6\,\|\varphi\|_\infty^2}{N}.$$

Proof.

$$\gamma_T^N(\varphi) - \gamma_T(\varphi) = \frac{1}{\sqrt{N}}\int_0^T p_{t-}^N \; d\mathbb{M}_t + \frac{1}{\sqrt{N}}\int_0^T p_{t-}^N \; d\mathcal{M}_t + \gamma_0^N(Q^T\varphi) - \gamma_0(Q^T\varphi),$$

(i) Initial condition is OK by independence.

L^2 estimates

(ii) $\mathcal{M}$-terms. Using Ito's isometry and $d[\mathcal{M},\mathcal{M}]_t \leqslant 4\|\varphi\|_\infty^2 d\mathcal{N}_t$, we obtain

$$\mathbb{E}\left[\left(\int_0^T p_{t-}^N \, d\mathcal{M}_t\right)^2\right] = \mathbb{E}\left[\int_0^T (p_{t-}^N)^2 d[\mathcal{M},\mathcal{M}]_t\right]$$

$$\leqslant 4\|\varphi\|_\infty^2 \frac{1}{N}\sum_{j=1}^{\infty}(1-\tfrac{1}{N})^{2(j-1)} \leqslant 4\|\varphi\|_\infty^2.$$

(iii) $\mathbb{M}$-terms. In the same way, applying Ito's isometry and the 'key formula' $p_{t-}^N d(\mathbb{M},\mathbb{M})_t = d\gamma_t^N(Q^2) + \text{Martingale}$, we get

$$\mathbb{E}\left[\left(\int_0^T p_{t-}^N \, d\mathbb{M}_t\right)^2\right] = \mathbb{E}\left[\int_0^T (p_{t-}^N)^2 d[\mathbb{M},\mathbb{M}]_t\right]$$

$$\leqslant \mathbb{E}\left[\int_0^T p_{t-}^N \, d(\mathbb{M},\mathbb{M})_t\right] = \mathbb{E}\left[\gamma_T^N(Q^2)\right] \leqslant \|\varphi\|_\infty^2.$$

INgredient (iii): Time uniform a priori estimate of p_t^N

Lemma

One has

$$\sup_{t\in[0,T]}\left|p_t^N - p_t\right| \xrightarrow[N\to\infty]{\mathbb{P}} 0.$$

Proof. Independent of the context. $t \mapsto p_t$ is continuous on $[0,T]$ by construction, it is cleat that $t \mapsto p_t^N$ is decreasing for all $N \geqslant 2$. The Lemma results of last Proposition and a from a probabilistic version of Second Dini (or Pólya) theorem: if a sequence of monotone functions converges pointwise on a compact interval and if the limit function is also continuous, then the convergence is uniform on that interval.

Increasing Process in general CLT

- In order to use martingale CLT, we need a càdlàg increasing process i_t^N such that $(\gamma^N(Q)_t)^2 - i_t^N$ is a martingale (quadratic variation - like).
- After tedious computations and trials and errors we chose a quadratic variation with only the branching jumps integrated.

$$i_t^N = \int_0^t \left(p_{u-}^N\right)^2 d(\mathbb{M},\mathbb{M})_u - \int_0^t \mathbf{Var}_{\eta_{u-}^N}(Q) p_{u-}^N \, dp_u^N + \frac{1}{N}\int_0^t \left(p_{u-}^N\right)^2 d\mathcal{R}_u .$$

- NB: with the rest term is $O(1/N)$ with

$$\mathcal{R}_t = \sum_{n=1}^{N}\sum_{k=1}^{+\infty}\left((1-\tfrac{1}{N})^2 \, \mathbf{Var}_{\eta^{(n)}_{\tau^-_{n,k}}}(Q) - \mathbf{Var}_{\eta^N_{\tau^-_{n,k}}}(Q) \right) 1_{t \geqslant \tau_{n,k}} .$$

Integration by parts formulas

Let $t \mapsto z_t^N$ be any càdlàg semi-martingale, $c > 0$ a deterministic constant, and assume for any branching time τ_j, $j \geqslant 1$:

- If $|\Delta z_{\tau_j}^N| \leqslant c/N$,then

$$\int_0^t p_{s-}^N \, dz_s^N = p_t^N z_t^N - z_0^N - \int_0^t z_{s-}^N \, dp_s^N + O(1/N).$$

- If $|z_{\tau_j^-}^N| \leqslant c(1-1/N)^j$,

$$\int_0^t z_{s-}^N \left(p_{s-}^N\right)^{-1} dp_s^N = \int_0^t z_{s-}^N \, d \ln p_s^N + O(1/N).$$

- If $|\Delta z_{\tau_j}^N| \leqslant c(1-1/N)^j/N$,

$$\int_0^t z_{s-}^N \, d \ln p_s^N = z_t^N \ln p_t^N - \int_0^t \ln p_{s-}^N \, dz_s^N + O(1/N).$$

Integration by parts for i_t^N

Using the intergation by parts formula abvoe and the key formula :

Lemma

The increasing process i_t^N can be integrated by parts and be rewritten as

$$i_t^N = p_t^N \gamma_t^N(Q^2) - \gamma_0^N(Q^2) + \left[\gamma_t^N(Q)\right]^2 \ln p_t^N - 2\int_0^t \gamma_{u^-}^N(Q^2)dp_u^N + O(\frac{1}{\sqrt{N}}).$$

Convergence of i_t^N and final proof of the CLT

By the non synchronous jump Assumption all the vanishing jump assumptions in the martingale CLT are verified.
The only remaining remaining part to prove is the following:

Proposition

For any $t \in [0, T]$, one has

$$i_t^N \xrightarrow[N\to\infty]{\mathbb{P}} i_t(\varphi). \qquad \textit{where}$$

$$i_t^N = p_t^N \gamma_t^N(Q^2) - \gamma_0^N(Q^2) + \left[\gamma_t^N(Q)\right]^2 \ln p_t^N - 2\int_0^t \gamma_{u^-}^N(Q^2)\,dp_u^N + O\left(\frac{1}{\sqrt{N}}\right.$$

$$i_t(\varphi) = p_t\gamma_t(Q^2) - \gamma_0(Q^2) + [\gamma_t(Q)]^2 \ln p_t - 2\int_0^t \gamma_u(Q^2)\,dp_u.$$

Proof of Convergence of i_t^N

After some calculations, all but one term can be treated using the L^2 a priori convergence estimate. The only remaing problem is the following:

$$i_t^N - i_t(\varphi) = \text{easy converging terms with } L^2\text{-estimate and IPP} +$$
$$-2\int_0^t (p_{u^-}^N - p_u)\, d\,\gamma_u^N(Q^2)$$

It is difficult to prove its convergence to 0 because the L^2 estimate is only pointwise. Hence handling the integrator is cumbersome !!.

Convergence of i_t^N

This difficult term is then treated as follows. Use the 'key formula'

$$\int_0^t (p_{u^-}^N - p_u)\, d\,\gamma_u^N(Q^2) = \int_0^t (p_{u^-}^N - p_u) p_{u^-}^N \, d(\mathbb{M},\mathbb{M})_u + O(\frac{1}{\sqrt{N}})$$

Since $(\mathbb{M},\mathbb{M})$ is an increasing process, it comes

$$\left| \int_0^t (p_{u^-}^N - p_u) p_{u^-}^N \, d(\mathbb{M},\mathbb{M})_u \right| \leqslant \sup_u |p_{u^-}^N - p_u| \times \left(\int_0^t p_{u^-}^N \, d(\mathbb{M},\mathbb{M})_u \right).$$

- The 'key formula' back again implies:

$$\mathbb{E}\left[\int_0^t p_{u^-}^N \, d(\mathbb{M},\mathbb{M})_u \right] = \mathbb{E}\left[\gamma_t^N(Q^2) \right] \leqslant \|\varphi\|_\infty^2.$$

- The a priori uniform estimate implies convergence of $\sup_u |p_{u^-}^N - p_u|$.

Arxiv

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