Ambit Fields & Atmospheric Turbulences

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Abstract

My PhD research concerns the connections between probability and physics. It regards the development and extension of mathematical models and theories finalised to understand physical systems, in particular atmospheric turbulences. In this poster I introduce turbulences, their mathematical properties, a mathematical theory developed to understand them (called ambit field theory) and some new results.

What is a turbulence?

Turbulence or turbulent flow is a flow regime in fluid dynamics characterized by chaotic changes in pressure and flow velocity. Turbulence is caused by excessive kinetic energy in parts of a fluid flow, which overcomes the dampening effect of the fluid's viscosity. For this reason turbulence is easier to create in low viscosity fluids, but more difficult in highly viscous fluids [1].

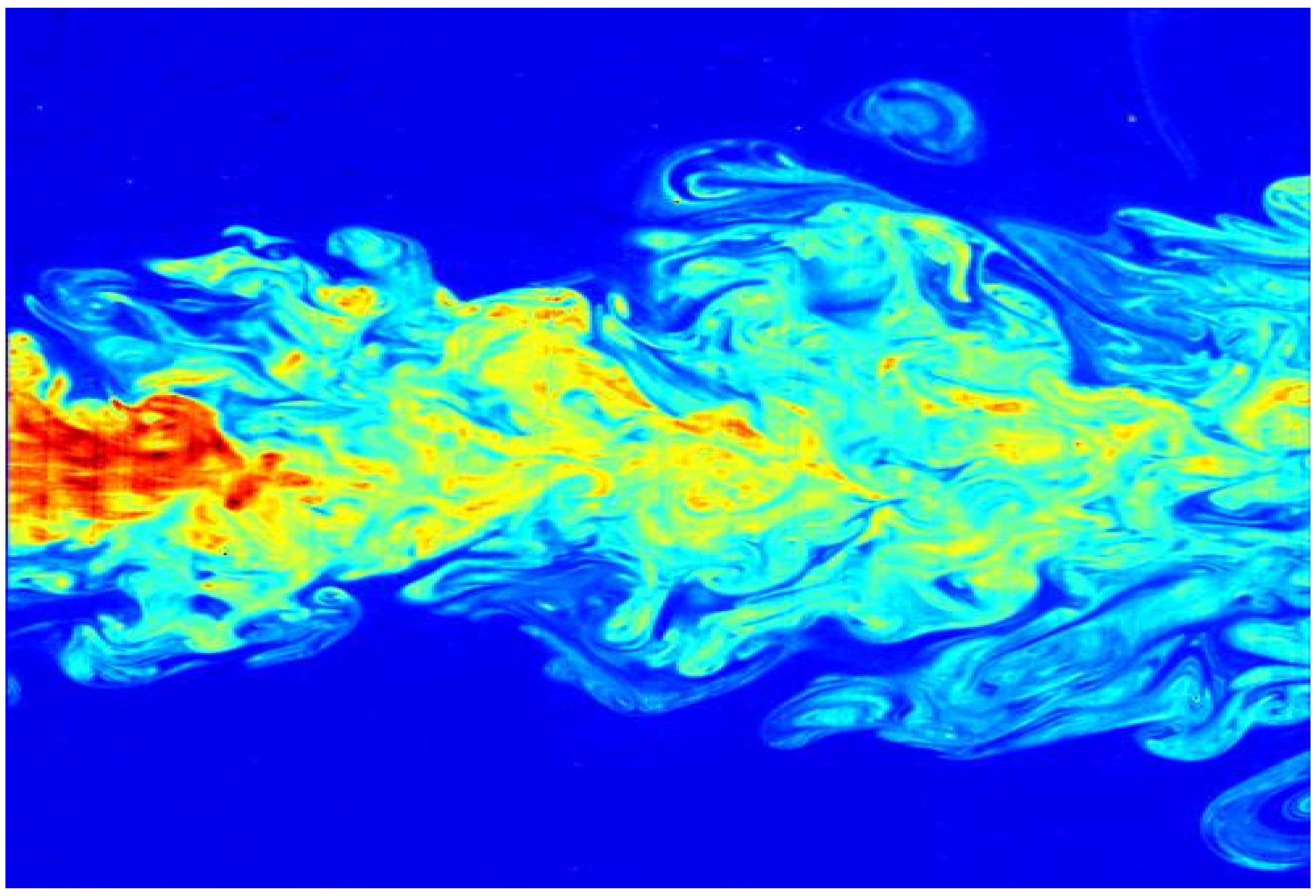

Figure : Flow visualization of a turbulent jet (source: Wikipedia).

What is an ambit field?

An ambit field is a d-dimensional random field describing the stochastic properties of a given system. More precisely, a tempo-spatial $\mathbb{R}^d$-valued ambit field, X, is a random field in space-time $\Omega \times \mathbb{R}$ taking values in $\mathbb{R}^d$ (usually $\Omega = \mathbb{R}^k$, $k \in \mathbb{N}$) such that

$$X_t(x) = \alpha + \int_0^t \int_\Omega g(\xi, s; x, t)\sigma_s(\xi)L(d\xi, ds) + \int_0^t \int_\Omega h(\xi, s; x, t)a_s(\xi)d\xi ds$$

where $\alpha \in \mathbb{R}$, g and h are deterministic multivariate functions, σ and a are random fields and L is a multivariate Lévy basis (see the definition of a Lévy basis in the section *Example: the Mixed Moving Average*).

Why ambit fields are useful to understand turbulences

Stochastic partial differential equations (SPDEs) constitute the main mathematical tool to model the dynamics of a random process, or field. In some cases, solutions of SPDEs are given as a stochastic integral of a Green's function convolved with another function. Therefore, it is possible to model the field of interest directly through a stochastic integral, taking a similar form as a solution through a Green's Function, instead of first specifying an SPDE and then trying to find a solution to this. This is the core idea of ambit fields and it provides a very flexible and general framework for modelling a variety of phenomena [2]. This is particularly important for turbulence and, in general, for any random system exhibiting high volatility, because thanks to their flexibility ambit fields can handle and model more accurately the behaviour of these systems. Indeed, ambit fields were initially developed to understand turbulences (see [1]).

Three mathematical properties of turbulences

Stationarity: An $\mathbb{R}^d$ random field $(X_t)_{t\in\mathbb{R}^l}$ is (strictly) stationary if its finite dimensional distribution (FDD) is invariant under translation of its parameters. In other words, let $F_{t_1,\dots,t_n}(x_1,\dots,x_n)$ be the FDD of $(X_t)_{t\in\mathbb{R}^l}$ for parameters $t_1,\dots,t_n \in \mathbb{R}^l$ and points $x_1,\dots,x_n \in \mathbb{R}^d$ then for any $s \in \mathbb{R}^l$

$$F_{t_1,\dots,t_n}(x_1,\dots,x_n) = F_{t_1+s,\dots,t_n+s}(x_1,\dots,x_n).$$

Infinite divisibility: An $\mathbb{R}^d$ random field $(X_t)_{t\in\mathbb{R}^l}$ is infinitely divisible if for every $k, n \in \mathbb{N}$ we have that $(X_{t_1},\dots,X_{t_k}) = Y_1 + \cdots + Y_n$, where $Y_i, i = 1,\dots,n$ are i.i.d. random vectors in $\mathbb{R}^{d\times k}$ (possibly on a different probability space). Alternatively, if all the FDDs of $(X_t)_{t\in\mathbb{R}^l}$ are infinitely divisible, which means that for every $k, n \in \mathbb{N}$ we have that $F_{t_1,\dots,t_k}(x_1,\dots,x_k) = \overset{(n \text{ times})}{\mu_n * \cdots * \mu_n} = \mu_n^{*n}$, where μ_n is a probability measure on $\mathbb{R}^{d\times k}$.

Mixing: A stationary $\mathbb{R}^d$ random field $(X_t)_{t\in\mathbb{R}^l}$ is mixing if

$$\lim_{n\to\infty} \mathbb{P}(A \cap \theta_{t_n}(B)) = \mathbb{P}(A)\mathbb{P}(B) \qquad \forall A, B \in \sigma(X) \text{ and } \forall (t_n)_{n\in\mathbb{N}} \in \mathcal{T},$$

where θ_t is a measure preserving $\mathbb{R}^d$-action such that $X_t(\omega) = X_0 \circ \theta_t(\omega)$, $\sigma(X)$ is the σ-algebra generated by $(X_t)_{t\in\mathbb{R}^l}$ and $\mathcal{T} := \left\{(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^l : \lim_{n\to\infty} \|t_n\|_\infty = \infty\right\}$.

For references about the relation of these properties with turbulences see [1],[2],[3],[5], among many.

New main result

The following theorem is an extension to the multivariate random field case of the celebrated result of Maruyama [4].

Theorem 1. Let $(X_t)_{t\in\mathbb{R}^l}$, with $l \in \mathbb{N}$, be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field such that Q_0, the Lévy measure of $\mathcal{L}(X_0)$, satisfies $Q_0(\{x = (x_1,\dots,x_d)' \in \mathbb{R}^d : \exists j \in \{1,\dots,d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i(X_{t_n}^{(j)} - X_0^{(k)})}\right] = \mathbb{E}\left[e^{iX_0^{(j)}}\right] \cdot \mathbb{E}\left[e^{-iX_0^{(k)}}\right]$$

for any $j, k = 1,..,d$ and for any sequence $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.

Example: the Mixed Moving Average

Definition *(Lévy basis)*. Let $l \in \mathbb{N}$ and $\mathcal{B}(\mathbb{R}^l)$ be the Borel σ-algebra of $\mathbb{R}^l$. Define $\mathcal{B}_b(\mathbb{R}^l) := \{A \in \mathcal{B}(\mathbb{R}^l) : Leb(A) < \infty\}$. A family of random vectors $(\Lambda(A) : a \in \mathcal{B}_b(\mathbb{R}^l))$ of random vectors in $\mathbb{R}^d$ is called a Lévy basis on $\mathcal{B}_b(\mathbb{R}^l)$ if:

1) The law of $\Lambda(A)$ is infinitely divisible for any $A \in \mathcal{B}_b(\mathbb{R}^l)$.

2)If $A_1, A_2, \dots, A_n \in \mathcal{B}_b(\mathbb{R}^l)$ are disjoint, then $\Lambda(A_1), \Lambda(A_2), \dots, \Lambda(A_n)$ are independent.

3) If $A_1, A_2, \dots \in \mathcal{B}_b(\mathbb{R}^l)$ are disjoint with $\bigcup_{i=1}^\infty A_i \in \mathcal{B}_b(\mathbb{R}^l)$, then $\Lambda(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \Lambda(A_i)$, a.s.

Theorem 2. Let

$$X_t = \int_S \int_{\mathbb{R}^l} f(A, t-s)\Lambda(dA, ds), \qquad t \in \mathbb{R}^l,$$

be a Mixed Moving Average random field where Λ is an $\mathbb{R}^d$-valued Lévy basis on $S \times \mathbb{R}^l$ and $f : S \times \mathbb{R}^l \to M_{q\times d}(\mathbb{R})$ is a measurable function. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing.

References

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