Data assimilation in the 2D Navier Stokes equations

Lea Oljaca,
Supervisors: Jochen Broecker, Tobias Kuna, Peter Jan van Leeuwen, University of Reading

EPSRC Centre for Doctoral Training
Mathematics of Planet Earth
Imperial College London
University of Reading

Introduction

Key question: Does data assimilation lead to improvements in the estimate of initial state? Is the error in the observations contained?

We study data assimilation for the 2D viscous, incompressible Navier-Stokes (N-S) equations, with L-periodic boundary conditions as a simpler proxy for the full governing equations.

$$\frac{\partial U}{\partial t} - \nu \Delta U + (U.\nabla)U + \nabla p = f, \qquad (1)$$
$$\nabla . U = 0,$$

where U is the velocity, ν is the kinematic viscosity, f the time independent body forcing and p the pressure.

Navier-Stokes equations

We assume that the model is a perfect representation of the atmosphere, so that a solution $U(t)$ represents the true weather. We can express U in terms of its Fourier series

$$U = \sum_{\bar{k}} U_{\bar{k}} e^{i\bar{k}.x},$$

where $\bar{k}$ is a 2-D non-zero integer vector.

The N-S equations (1) can be reformulated in Fourier space, while the zero divergence condition and periodicity allows us to remove the pressure term. In this formulation, (1) becomes

$$\frac{dU}{dt} + \nu AU + B(U, U) = f \qquad (5)$$

which is an ODE and A and B are operators s.t. $(AU, v) = \int_\Omega -\Delta U.v \, dx$, $(B(U, U), v) = \int_\Omega (U.\nabla U).v \, dx$.

A model for observations

An observation of the weather at time t_n is given by $P_\lambda U(t_n) + \sigma R_n$, where σR_n models the error in the observation.

We suppose that we can observe Fourier modes of the solution and we define a projection P_λ onto the observation space given by

$$P_\lambda U = \sum_{|\bar{k}|^2 \leq \lambda} U_{\bar{k}} e^{i\bar{k}.x},$$

where λ is a finite positive integer. We say that P_λ is a projection onto the 'low' modes and λ is a measure of the size of the observed space or the largest wave number still observed.

A discrete data assimilation algorithm

Straightforward replacement of model values by observations.

The approximating solution of discrete data assimilation is obtained by inserting the observations at discrete times t_n such that

$$u_0 = \eta + P_\lambda U_0 + \sigma R_0$$

and

$$u_n = Q_\lambda \psi(t_n, t_{n-1}, u_{n-1}) + P_\lambda U(t_n) + \sigma R_n, \qquad (2)$$

where ψ is the semi-flow of (1), η is the initial guess and $Q_\lambda = I - P_\lambda$ is the projection onto the unobserved space. Then, the approximating solution $u(t)$ is a piece-wise continuous in time function defined by

$$u_n(t) = \psi(t, t_n, u_n) \text{ for } t \in [t_n, t_{n+1}). \qquad (3)$$

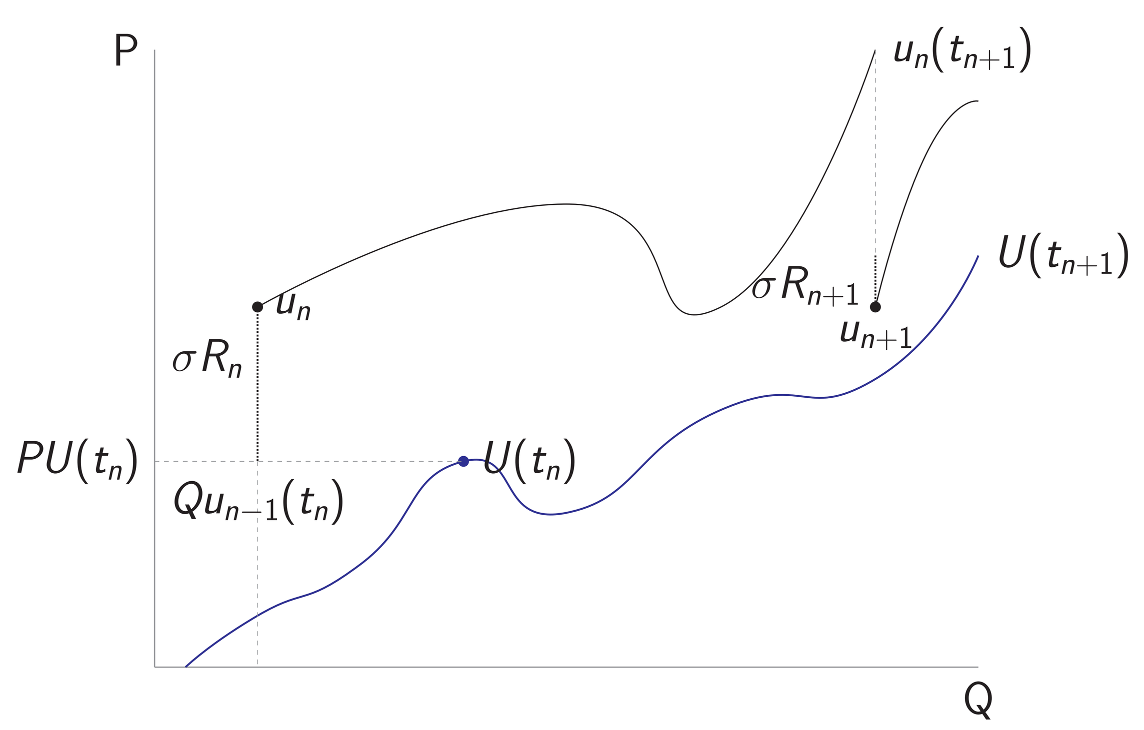

Figure: Illustration of data assimilation method with observation error R_n

Data assimilation error

The data assimilation error $\delta(t)$ is the difference between the true and approximating solution. It is a piece-wise continuous in time function defined by

$$\delta_n(t) = U(t) - u_n(t) \qquad (4)$$

in the interval $[t_n, t_{n+1})$.

Main Theorem

We are able to prove a rigorous estimate derived using analytical properties of the underlying dynamics.

Theorem Let U be as defined by (5) and δ_n be the data assimilation error as defined by (4) and suppose that $\mathbb{E}(\|R_n\|^{8/3}) < \infty$. Then, for any data assimilation interval $h = t_{n+1} - t_n > 0$, $\exists$ a finite $\lambda^*(h)$ such that for all $\lambda > \lambda^*$,

$$\limsup_{n \to \infty} \left(\|\delta_n\|^2 - \sigma^2 B_n \right) \leq 0$$

a.s., where B_n is a stationary a.s. finite process and $\sigma^2 B_n \to 0$ as $\sigma \to 0$ a.s.

Main assumptions

Noise: We assume that the observation error is random, unbounded and R_n is a stationary, tempered process with zero mean and $\mathbb{E}(|R_n|^2) = 1$. Therefore $\mathbb{E}(|\sigma R_n|^2) = \sigma^2 < \infty$ and σ^2 is the variance of observation noise.

We note that if R_n does not have zero mean, this would represent a systematic error, which would likely be corrected for and therefore we can make the simplifying assumption.

Model: There is no model error. That is, the dynamical system (1) is a perfect representation of the atmosphere and we use it for the forecasting.

Observations: We can observe the 'low' Fourier modes of the true solution.

Proof part I: The error equation

Using the bi-linearity of B and the fact that the data assimilated solution satisfies the equation in every interval $[t_n, t_{n+1})$ we obtain the error equation;

$$\frac{d\delta}{dt} + \nu A\delta + B(U, \delta) + B(\delta, U) - B(\delta, \delta) = 0. \qquad (6)$$

Using the above, as well as estimates as in [1] and iterating over the intervals $[t_n, t_{n+1})$, we obtain;

Lemma 1

$$\|\delta(t_{n+1})\|^2 \leq M_n(h)\|\delta(t_n)\|^2 + \sigma^2 \|R_{n+1}\|^2, \qquad (7)$$

where $\|.\|$ is the H^1 norm and $M_k(h)$ are functions that depend on $\delta(t_k)$.

Proof part II: Controlling the error

To obtain a meaningful bound on the error $\|\delta(t_{n+1})\|^2$, we would need that RHS of (7) is almost surely finite in the long term. This will be the case if we can make $M_k < 1$ for all k.

Since $\delta(t_k)$ are stochastic, the M_k are also stochastic and therefore it is not, in general, possible to guarantee that $M_k < 1$ for all k for any value of h. However, we are able to use the Ergodic Theorem to show that if $\mathbb{E}(M_k) < 1$, it ensures that $M_k < 1$ 'often enough'. That is, for almost all realizations of the sequence $\{M_k\}_k$, the proportion of $M_k < 1$ is sufficient to ensure that the error remains almost surely finite.

References

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Lea.Oljaca@pgr.reading.ac.uk