Stability of a frontogenesis slice model

Aythami Bethencourt de Leon, Imperial College London

EPSRC Centre for Doctoral Training

Mathematics of Planet Earth

Imperial College London

University of Reading

Introduction to the slice model.

In the Cotter-Holm Slice Model (intended as a model for atmospheric simulation), one considers a dynamical system with Lagrangian map of the form

$$\phi(X,Y,Z,t) = (x(X,Z,t), y(X,Z,t) + Y, z(X,Z,t)), \quad (X,Z) \in \Omega \subset \mathbb{R}^2, Y \in \mathbb{R}. \tag{1}$$

Here, Ω is called the slice and $Y \in \mathbb{R}$ is thought as an element in the transverse component to the slice. Note that (1) represents the map from a current configuration of particles (X,Y,Z) at time zero, to a future configuration (x,y,z) at time t. This type of deformation map gives raise to two velocities which play a role in the slice model:

- The two-dimensional velocity on the slice $u_S \equiv u_S(x,z)$,
- The one-dimensional transverse velocity $u_T \equiv u_T(x,z)$.

Conservation of mass in the slice model reads

$$\frac{\partial D}{\partial t} + \nabla\cdot(u_S D) + \frac{\partial(u_T D)}{\partial y} = 0, \tag{2}$$

where D denotes the mass density. Since D and u_T are assumed to be $y-$independent, (2) can be reformulated as

$$\frac{\partial D}{\partial t} + \nabla\cdot(u_S D) = 0.$$

In this model, potential temperature is defined by

$$\theta(x,y,z,t) = \theta_S(x,z,t) + (y-y_0)s,$$

for a constant s (tracking the degree of linear variation of θ_S in the y component), and the tracer equation for the potential temperature is

$$\frac{\partial\theta_S}{\partial t} + u_S\cdot\nabla\theta_S + u_T s = 0.$$

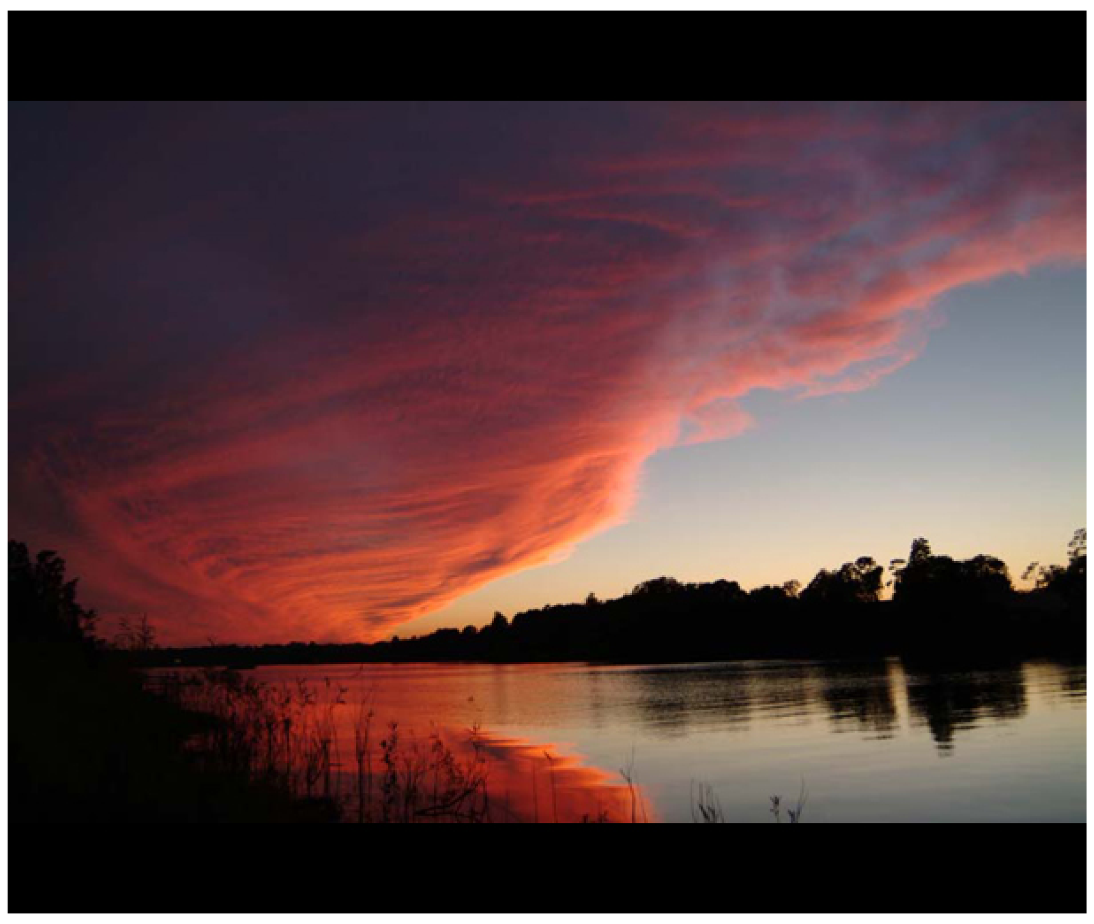

Figure : Formation of a cold front.

Slice models are frequently employed to understand atmospheric dynamics. In particular, they are useful to understand the process of frontogenesis.

Slice model equations.

To obtain the equations for the slice model, one considers a Lagrangian function of the type

$$l \equiv l[(u_S,u_T),(\theta_S,s),D],$$

which is a smooth function depending on the variables written above. The slice model equations are the trajectories of minimal energy associated with that Lagrangian. This is, we consider the action functional

$$S[(u_S,u_T),(\theta_S,s),D] = \int_0^T l[(u_S,u_T),(\theta_S,s),D]\,dt,$$

where the last integral is taken over closed paths for the variables, vanishing at the endpoints. The equations are obtained after applying Hamilton's principle

$$\delta S = \delta\int_0^T l[(u_S,u_T),(\theta_S,s),D]\,dt = 0.$$

Incompressible Euler-Boussinesq case.

The incompressible Euler-Boussinesq Eady model in a periodic domain $(x,z)\in\Omega$ of width L and height H has Lagrangian

$$l = \int_\Omega\left\{\frac{D}{2}(|u_S|^2+|u_T|^2) + Dfu_T x + \frac{g}{\theta_0}D\left(z-\frac{H}{2}\right)\theta_S + p(1-D)\right\}dV. \tag{3}$$

Here dV stands for integration over the slice domain Ω. The constant f is the Coriolis parameter, and p is just a multiplier which ensures $D=1$, implying $\nabla\cdot u_S = 0$ (incompressibility). Note that instead of choosing (3), we might have chosen any Lagrangian with physical significance. The equations for the incompressible slice model are obtained to be

$$\begin{aligned}\frac{\partial u_S}{\partial t} + u_S\cdot\nabla u_S - fu_T\hat{x} &= -\nabla p + \frac{g}{\theta_0}\theta_S\hat{z},\\ \frac{\partial u_T}{\partial t} + u_S\cdot\nabla u_T + fu_S\cdot\hat{x} &= -\frac{g}{\theta_0}\left(z-\frac{H}{2}\right)s,\\ \frac{\partial\theta_S}{\partial t} + u_S\cdot\nabla\theta_S + u_T s &= 0,\\ \nabla\cdot u_S &= 0.\end{aligned} \tag{4}$$

Conserved quantities for the incompressible Euler-Boussinesq case.

The incompressible Euler-Boussinesq equations conserve energy and generalised enstrophy:

$$h = \int_\Omega\left\{\frac{1}{2}|u_S|^2 + \frac{1}{2}u_T^2 - \gamma_S\theta_S\right\}dV, \quad \text{(Energy)} \tag{5}$$

$$C_\Phi = \int_\Omega \Phi(q)\,dV, \quad \text{(Generalised enstrophy)} \tag{6}$$

for any differentiable function Φ of the potential vorticity $q := curl(v_S)\cdot\hat{y}$, $v_S = su_S - (u_T + fx)\nabla\theta_S$. Conserved quantities (5) and (6) are fundamental for understanding the system and for studying stability of equilibria.

Equilibrium solutions for the incompressible Euler-Boussinesq case.

Stationary solutions of the incompressible Euler-Boussinesq equations are critical points of the generalised Hamiltonian

$$H_\Phi = \int_\Omega\left\{\frac{1}{2}(|u_S|^2 + u_T^2) - \gamma_S\theta_S\right\}dV + \int_\Omega \Phi(q)\,dV.$$

These are given by the conditions

$$\begin{aligned}a_i &= \Phi'(q_e|\partial\Omega_i), \;\; i = 0,\dots,n,\\ u_{Se} &= -curl(\Phi'(q_e)\hat{y})s,\\ u_{Te} &= curl(\Phi'(q_e)\hat{y})\cdot\nabla\theta_{Se},\\ \gamma_S &= curl(\Phi'(q_e)\hat{y})\cdot(\nabla u_{Te} + f\hat{x}).\end{aligned}$$

For this, we need the natural no-slip condition

$$u_S\cdot\hat{n} = 0$$

on the boundary. Note that sub-indices "e" are used to denote equilibrium states.

Monge-Ampere equation for the slice model.

The geopotential $\phi(x,z)$ is a function on the slice satisfying

$$\nabla\phi(x,z) = \frac{g}{\theta_0}\theta_S\hat{z} + fu_T\hat{x}.$$

The equality

$$q_R = -curl((u_T+fx)\nabla\theta_S) = det(D^2(\phi + (fx)^2/2))$$

is a Monge-Ampere equation on the function $\phi + (fx)^2/2$ (if we consider q_R as a function of the spatial position $q_R(x)$). This equation is elliptic when $\phi + (fx)^2/2$ is a convex function. Exactly as in the semigeostrophic model, when the Monge-Ampere equation loses its ellipticity a front will form.
Note that the conserved vorticity q for the slice model is the sum of two terms: $\omega_S = scurl(u_S)$ and q_R, which is responsible for front generation.

Stability conditions for the incompressible Euler-Boussinesq model.

An equilibrium point of the incompressible slice equations belonging to the specified restricted class of equilibria is formally stable if $\Phi''(q_e) > 0$. This is, an equilibrium point is formally stable if

$$\frac{(\hat{y}\times\nabla q_e)\cdot u_{Se}}{s|\nabla q_e|^2} > 0.$$

With a few more not very restrictive conditions we can obtain nonlinear stability. Note that this theorem generalizes Rayleigh-Arnold first theorem to the slice model case (which adds potential temperature and a transversal velocity to the model). Rayleigh-Arnold second theorem (formal stability conditions when $\Phi''(q_e) < 0$) is however broken in the slice model! Fronts are associated to discontinuities in the potential temperature and vorticity. Therefore, when a front forms, $\nabla q_R = \infty$ and nonlinear stability is completely lost.

Main references

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ab1113@ic.ac.uk