Conservative numerical schemes for wave equations

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Problem

Wave equations are very useful tools, and can be viewed as a simplification of the Navier-Stokes equations. Under various assumptions the Navier-Stokes equations can be used to derive the shallow water equations, which in turn can be used to derive both wave equations we will consider. The equations we consider are one dimensional PDEs, i.e., $u = u(t, x)$. We consider a temporal domain of $t \in [0, T]$ for $T \in \mathbb{R}^+$ and a periodic spacial domain of $\Omega = [x_l, x_r]$ for $x_l, x_r \in \mathbb{R}$. The first is the Korteweg-de Vries *(KdV)* equation, and is derived via asymptotic expansions of the shallow water equations. One physical application is the prediction of tsunamis. The KdV equation is described by

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0,$$

and it omits travelling wave solutions known as solitons. These solitons are linear combinations of

$$u(t, x) = \frac{1}{2} c \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - p)\right),$$

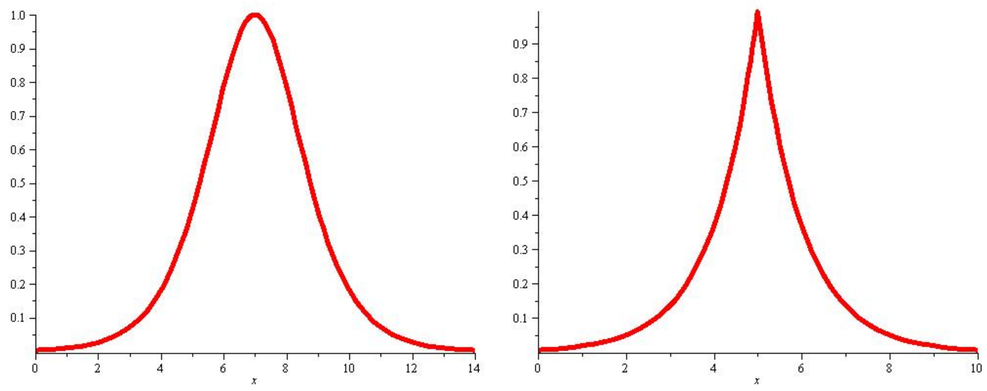
where c denotes wavespeed, and p denotes the initial position of the soliton. The second is the Camassa-Holm *(CH)* equation and is derived similarly to KdV but the asymptotic expansion is one order higher. The CH equation is given by

$$\partial_t u + 2\kappa\partial_x u - \partial_{xxt} u + 3u\partial_x u - 2\partial_x u\partial_{xx} u - u\partial_{xxx} u = 0,$$

where κ is the dispersion parameter. For $\kappa > 0$ we obtain soliton solutions. In the case $\kappa = 0$ we obtain peakon solutions, which feature a nondifferentiable point. These peakons are described by linear combinations of

$$u(t, x) = c \exp\left(-|x - ct - p|\right),$$

where c is the initial amplitude of the peakon and p denotes the initial position.



(a) Soliton with $c = 2$ and $p = 7$. (b) Peakon with $c = 1$ and $p = 5$.

Figure: The initial state of the travelling wave solutions in KdV (soliton) and CH with $\kappa = 0$ (peakon). Note that multiple soliton/peakon solutions can be investigated simply by addition of more solitons/peakons in the initial conditions. Additionally notice that peakons contain a $\mathcal{C}^0$ point, this highlights that peakon solutions solve CH in the weak sense.

Conserved quantities

Both KdV and CH are integrable systems, so they have infinitely many conserved quantities. For KdV we focus on mass and energy conservation, i.e.,

$$\partial_t \int_\Omega u = 0 \qquad \text{and} \qquad \partial_t \int_\Omega u^3 - \frac{1}{2}(\partial_x u)^2 = 0$$

respectively. We are also interested in mass and an energy conservation for CH, which are given by

$$\partial_t \int_\Omega u = 0 \qquad \text{and} \qquad \partial_t \int_\Omega u^3 - (u - 2\kappa)(\partial_x u_x)^2 = 0$$

respectively. We want to conserve discrete counterparts of these properties, the conservation of these physically relevant discrete quantities ensures a physically realistic solution, as well as guaranteeing numerical stability.

Notation

Our conservative numerical schemes are discontinuous Galerkin *(dG)* methods. These methods are comprised of piecewise polynomial trial functions tested against piecewise polynomial functions. Note that we are not enforcing global continuity on the piecewise polynomials. The key reason we choose dG methods is the discontinuities guarantee that derivatives are in our test space, which is required to mimic the proofs of energy conservation from the continuous level. For simplicity we partition our periodic domain $\Omega = [x_l, x_r]$ uniformly, i.e.,

$$x_l := x_0 < x_1 < \cdots < x_M =: x_r,$$

where $I_m = (x_{m-1}, x_m)$ are our elements and $x_m - x_{m-1} = h$ is our step size. Additionally we define $\mathbb{V}$ to be the space of polynomial functions over I_m of some arbitrary order. For simplicity consider linear polynomials. As we have discontinuities for our numerical methods to be accurate we need communication between the elements, this communication takes the form of jumps and averages over the nodes $\mathcal{E} := \{x_0, x_1, ..., x_M\}$, which are defined by

$$[\![U]\!]_m = \lim_{x \nearrow x_m} U - \lim_{x \searrow x_m} U \qquad \text{and} \qquad \{U\}_m = \frac{1}{2}\left(\lim_{x \nearrow x_m} U + \lim_{x \searrow x_m} U\right)$$

respectively.

A conservative numerical scheme

We wish to construct a scheme which conserves a discrete energy in KdV. To do this we introduce an auxiliary variable which is the first variation of the energy, i.e.,

$$v = 3u^2 + \partial_{xx} u,$$

which allows us to write KdV as the system

$$\begin{aligned} \partial_t u + \partial_x v &= 0, \\ v - 3u^2 - \partial_{xx} u &= 0. \end{aligned}$$

Applying an interior penalty dG method spatially to the system our numerical scheme is given by seeking $(U, V) \in \mathbb{V}^2$ such that

$$\begin{aligned} \int_\Omega (\partial_t U + \partial_x V)\,\phi + \int_\mathcal{E} [\![V]\!]\{\phi\} &= 0 \\ \int_\Omega \left(V - 3U^2\right)\psi + \mathcal{A}_h(U, \psi) &= 0 \qquad \forall (\phi, \psi) \in \mathbb{V}^2, \end{aligned}$$

where

$$\mathcal{A}_h(U, \psi) := \int_\Omega \partial_x U \partial_x \psi + \int_\mathcal{E} \left(-[\![U]\!]\{\partial_x \psi\} - [\![\psi]\!]\{\partial_x U\} + \frac{\sigma}{h}[\![U]\!][\![\psi]\!]\right).$$

This method conserves a discrete mass and energy, which are given by

$$M[U] := \int_\Omega U \qquad \text{and} \qquad E[U] := \int_\Omega U^3 - \frac{1}{2}\mathcal{A}_h(U, U)$$

respectively.

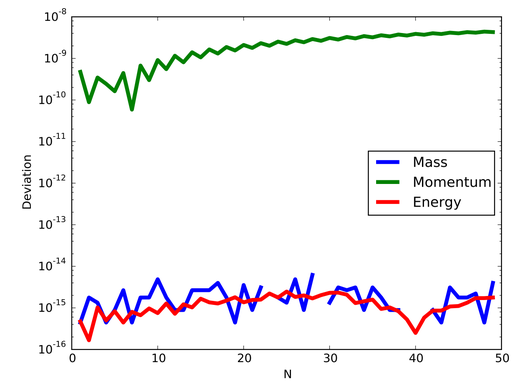


Figure: The deviation of mass, momentum and energy for a one soliton solution to KdV solved via the interior penalty dG method with a solver tolerance of 10^{-12}. We solve over 50 time steps (denoted N) and 400 nodes in space. Note that mass and energy are conserved, but momentum is not. While momentum is an important physical property we cannot develop a method conserving all three using our current methodology.

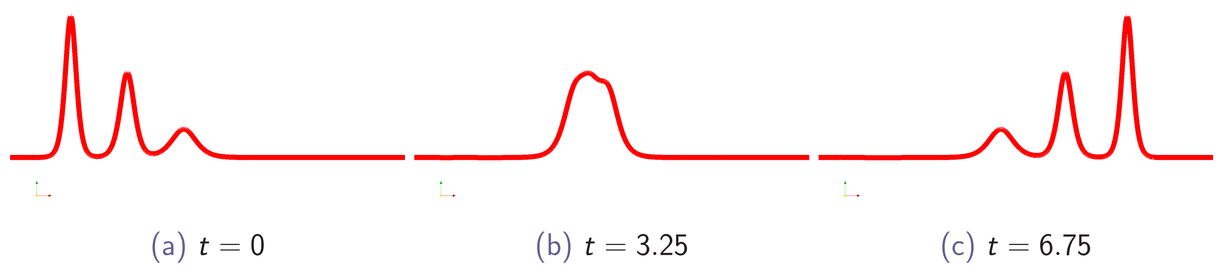
An a priori bound

As the interior penalty dG method conserves a discrete energy we can apply energy arguments to obtain an a priori error bound. The bound is given by

$$\begin{aligned} \frac{1}{2}\mathcal{A}_h(u - U, u - U) + \int_0^t \|\partial_x v - \partial_x V\|^2 \leq \exp(C_1 t) C_2 h^{2p} \int_0^t \left(|v|^2_{H^{p+1}} + |u|^2_{H^{p+1}}\right) \\ + C_3 h^{2p}\left(|u|^2_{H^{p+1}} + \int_0^t |v|^2_{H^{p+1}}\right), \end{aligned}$$

where $h \ll 1$ and $\sigma \geq 10$.

Soliton interactions



(a) $t = 0$ (b) $t = 3.25$ (c) $t = 6.75$

Figure: The dG interior penalty method for KdV with three solitons moving at different speeds. The solitons with larger amplitude overtake the smaller solitons and preserve their structure.

What next?

The next step is for us to construct an energy conserving method for the CH equation. We will do this by introducing the first variation of the energy as an auxiliary variable, i.e.,

$$v = 3u^2 - (\partial_x u)^2 - 2(u - 2\kappa)\partial_{xx} u,$$

allowing us to write CH as the system

$$\begin{aligned} \partial_t u + 2\kappa\partial_x u - \partial_{xxt} u + \frac{1}{2} v_x - 2\kappa\partial_{xxx} u &= 0 \\ v - 3u^2 + (\partial_x u)^2 + 2(u - 2\kappa)\partial_{xx} u &= 0. \end{aligned}$$

The difficulty now comes from choosing a discretisation such that a discrete energy is conserved.

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