A hybrid all-scale finite-volume module for global NWP

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**Motivation:** Simulations of global weather at non-hydrostatic resolutions will become operational for NWP beyond 2020. Existing NWP models proficiently operate at hydrostatic scales and are not equipped to efficiently resolve convective motions where non-hydrostatic effects dominate.

**Approach:** The autonomous finite-volume module (FVM) supplementing the ECMWF's Integrated Forecasting System (IFS) with capabilities of cloud-resolving models and a toolbox of methods unavailable in spectral-transform based semi-implicit/semi-Lagrangian (SISL) models.
FVM design:

- Governing PDEs include the anelastic, pseudo-incompressible and fully compressible (acoustic and large-time steps) nonhydrostatic systems.

- All PDEs are cast in time-dependent curvilinear coordinates, thus enabling a range of coordinate systems (especially in the vertical), and 3D r adaptivity.

- Nonoscillatory forward-in-time integrators are implicit with respect to buoyant, rotational and acoustic modes of motion.

- Two engines of the integrators: MPDATA “high-resolution” advection, and bespoke robust preconditioned GCR solver for stiff elliptic problems.

- Discretisation in computational space: a hybrid of finite-volume unstructured edge/node based in the horizontal, and flux-form structured finite-difference based in the vertical.

- Flexibility of the unstructured meshes and co-location of dependent variables enable finite-volumes built around reduced Gaussian grids of the IFS, thus supporting both finite-volume and spectral-transform discretisations and seemingly inheriting an “equal area” parallelisation scheme of the IFS.
**Theoretical background:** nonhydrostatic PDEs (either anelastic, pseudo-incompressible or compressible) are cast in time-dependent curvilinear coordinates, to design consistent control-volume integrations for a range of physical domains.

\[
\frac{\partial G\psi}{\partial t} + \nabla \cdot (G\psi \mathbf{v}) = G\mathbf{v} \cdot \mathbf{R} \\
\frac{\partial G\mathbf{v}}{\partial t} + \nabla \cdot (G\mathbf{v} \mathbf{v}) = 0
\]

\( \mathbf{v} = \dot{x} \) not necessarily equal to \( \mathbf{u} \)

\( G(x, t) \) denotes the Jacobian

ILES semi-implicit forward-in-time (2 time level) integrators, admitting schemes implicit with respect to buoyant, rotational and acoustic modes \( \Rightarrow \) compressible Euler equations can be integrated with acoustic or soundproof time steps.

**consider an archetype problem (AP):**

\[
\frac{\partial G\Psi}{\partial t} + \nabla \cdot (G\Psi \mathbf{v}) = G\mathbf{R}
\]

**MPDATA based “high-resolution” forward-in-time semi-implicit integrators for fluids**

\[
\Psi_i^{n+1} = A_i \left( \tilde{\Psi}^n, \mathbf{V}^{n+1/2}, G^n, G^{n+1} \right) + 0.5\delta t R_i^{n+1},
\]

where

\[
\tilde{\Psi}^n \equiv \Psi^n + 0.5\delta t R^n.
\]
“Banach principle”, an important tool for systems with nonlinear right-hand-sides:

\[ \forall i \quad \Phi_{i}^{n+1}, \mu = \Phi_{i}^{*} + 0.5 \delta t R_{i}^{n+1}, \mu^{-1} \]

\[ \forall i \quad \Phi_{i}^{n+1} = \Phi_{i}^{*} + 0.5 \delta t R_{i}^{n+1} \]

\[ \| \Phi^{n+1, \mu} - \Phi^{n+1} \| = 0.5 \delta t \| R(\Phi^{n+1, \mu^{-1}}) - R(\Phi^{n+1}) \| \leq 0.5 \delta t \sup \| \partial R / \partial \Phi \| \| \Phi^{n+1, \mu^{-1}} - \Phi^{n+1} \| \]

Eulerian semi-implicit integrators ➔ ➔ ➔
Two key developments unifying soundproof and compressible PDEs while enabling their consistent large time step integrations:

\[
\frac{\partial \mathcal{G} \varrho}{\partial t} + \nabla \cdot (\mathcal{G} \varrho \mathbf{v}) = 0 ,
\]

\[
\frac{\partial \mathcal{G} \varrho \theta'}{\partial t} + \nabla \cdot (\mathcal{G} \varrho \theta' \mathbf{v}) = -\mathcal{G} \varrho \tilde{\mathcal{G}}^T \mathbf{u} \cdot \nabla \theta_e ,
\]

\[
\frac{\partial \mathcal{G} \varrho \mathbf{u}}{\partial t} + \nabla \cdot (\mathcal{G} \varrho \mathbf{v} \otimes \mathbf{u}) =
\]

\[
-\mathcal{G} \varrho \left( \Theta \tilde{\mathcal{G}} \nabla \varphi + g \gamma_B \frac{\theta'}{\theta_{b}} + \mathbf{f} \times (\mathbf{u} - \gamma_C \mathbf{u}_e) - \mathcal{M}'(\mathbf{u}, \mathbf{u}, \gamma_C) \right)
\]

**i) For all specific variables, mass continuity defines transportive momenta**

\[
\varrho_i^{n+1} = \mathcal{A}_i \left( \varrho^n, (\mathcal{G} \mathbf{v})^{n+1/2}, \mathcal{G}, \mathcal{G} \right) \quad \Rightarrow \quad \mathbf{V}^{n+1/2} = \left( \mathcal{G} \varrho \mathbf{v} \right)^{n+1/2}
\]

in the spirit of soundproof models

**ii) Bespoke NFT Helmholtz solver implied by compressible NFT integrators,**

\[
0 = -\delta t \left[ \frac{1}{\mathcal{G}} \nabla \cdot (\mathcal{G} \mathbf{v}) + \frac{1}{\gamma} \frac{\pi_e}{\varrho^* \pi_e} \nabla \cdot (\varrho^* \pi_e \mathbf{v}) - \frac{1}{\gamma} \frac{\pi_e}{\varrho^*} \nabla \cdot (\varrho^* \mathbf{v}) \right] - \beta (\varphi - \tilde{\varphi})
\]

derived from the evolutionary form of the gas law integrated with the NFT solver --- extension of the generalised Poisson solver for the soundproof systems.
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 , \\
\frac{\partial \rho \theta'}{\partial t} + \nabla \cdot (\rho \theta' \mathbf{v}) = -\rho \tilde{G}^T \mathbf{u} \cdot \nabla \theta_e , \\
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{u}) = -\rho \left( \tilde{\Theta} \tilde{G} \nabla \varphi + g \tilde{\gamma}_B \frac{\theta'}{\theta_b} + \mathbf{f} \times (\mathbf{u} - \tilde{\gamma}_C \mathbf{u}_e) - \mathcal{M}'(\mathbf{u}, \mathbf{u}, \tilde{\gamma}_C) \right)
\]

**semi-implicit ``acoustic'' scheme:**

\[
\rho_{i}^{n+1} = A_i \left( \rho^n, (\rho \mathbf{V})^{n+1/2}, \rho, \mathcal{G} \right) \quad \implies \quad \mathbf{V}^{n+1/2} = (\rho \mathbf{V})^{n+1/2}
\]

\[
\tilde{\theta}_i' = A_i \left( \tilde{\theta}', \mathbf{V}^{n+1/2}, \rho^n, \rho_{*n+1} \right) \\
\tilde{\mathbf{u}}_i = A_i \left( \tilde{\mathbf{u}}, \mathbf{V}^{n+1/2}, \rho^n, \rho_{*n+1} \right)
\]

\[
\rho_{*n}^{n+1} := \mathcal{G} \rho^n \quad \text{and} \quad \rho_{*n+1}^{n+1} := \mathcal{G} \rho^{n+1}
\]

\[
\nu = 1, ..., \tilde{N}_\nu \quad (\text{RE: Banach principle})
\]

\[
\theta_{i}^{\nu'} = \tilde{\theta}_i' - 0.5 \delta t \left( \tilde{G}^T \mathbf{u}^{\nu} \cdot \nabla \theta_e \right)_i
\]

\[
\mathbf{u}_{*i}^{\nu'} = \tilde{\mathbf{u}}_i - 0.5 \delta t \left( \Theta^{\nu-1} \tilde{G} \nabla \varphi^{\nu} + g \tilde{\gamma}_B \frac{\theta_{*}^{\nu}}{\theta_b} \right)_i - 0.5 \delta t \left( \mathbf{f} \times (\mathbf{u}^{\nu} - \tilde{\gamma}_C^{\nu-1} \mathbf{u}_e) - \mathcal{M}'(\mathbf{u}, \mathbf{u}, \tilde{\gamma}_C)^{\nu-1} \right)_i
\]

\[
\varphi_{i}^{\nu'} = c_p \theta_0 \left[ \left( \frac{R_d}{p_0} \rho^{n+1} \theta_{*}^{\nu-1} \right)^{R_{d/c_v}} - \pi_e \right]_i
\]

\[
\theta_{i}^{\nu'} = \left( \tilde{\theta}' - 0.5 \delta t \tilde{G}^T \mathbf{u}^{\nu} \cdot \nabla \theta_e + \theta_e \right)_i
\]

\[
\theta_{i}^{0} = A_i \left( \theta^n, \mathbf{V}^{n+1/2}, \rho^n, \rho_{*n+1} \right)
\]

**simple but computationally unaffordable; example**

8 days, surface $\theta'$, 128x64x48 lon-lat grid, 128 PE of Power7 IBM

CPI2, $2880 \ dt=300 \ s$, wallclock time=2.0 mns

CPEX, $432000 \ dt=2 \ s$, wallclock time=178.9 mns

semi-implicit ``large time step” scheme ➔
elliptic boundary value problems (BVPs):

Poisson problem in soundproof models relies on the mass continuity equation
\( \nabla \cdot (\varrho^* \mathbf{v}) = 0 \)

Because \( \mathbf{v} = \mathbf{G}^T u \), acting with \( \mathbf{G}^T \) on both sides of \( u'' = \ddot{u} - C \nabla \varphi'' \) and ...

\[
0 = -\frac{\delta t}{\varrho^*} \nabla \cdot (\varrho^* u'') = -\frac{\delta t}{\varrho^*} \nabla \cdot \left[ \varrho^* \left( \ddot{u} - \mathbf{G}^T C \nabla \varphi'' \right) \right]
\]

diagonally preconditioned Poisson problem for pressure perturbation
Helmholtz problems for large-time-step compressible models also rely on mass continuity equation:

Combine the evolutionary form of the gas law & mass continuity equation in the inhomogeneous AP

\[
\frac{d\pi}{dt} = -\gamma \pi \nabla \cdot \mathbf{u} \iff \frac{\partial \rho \pi}{\partial t} + \nabla \cdot (\rho \pi \mathbf{u}) = -\gamma \rho \pi \nabla \cdot \mathbf{u} \quad \text{where } \gamma \equiv \frac{R_d}{c_v}
\]

\[
\frac{\partial \rho^* \pi}{\partial t} + \nabla \cdot (\rho^* \mathbf{v} \pi) = -\gamma \rho^* \pi \frac{1}{G} \nabla \cdot (G \mathbf{v})
\]

\[
\pi^{n+1} = \hat{\pi} - \delta t \gamma \pi^{n+1} \frac{1}{G} \nabla \cdot (G \mathbf{v}^{n+1}) + \mathcal{O}(\delta t^2)
\]

\[
\hat{\pi} = \mathcal{A} \left( \pi^n, \mathbf{v}^{n+1/2}, \rho^*_n, \rho^*_{n+1} \right)
\]

\[
0 = -\frac{\delta t}{G} \nabla \cdot \left[ G \left( \tilde{\mathbf{v}} - \mathcal{G}^T \mathcal{C} \nabla \varphi^\nu \right) \right] - \beta (\varphi^\nu - \varphi^\dagger)
\]

\[
\beta \equiv \left[ \gamma (\varphi^\nu^{-1} + c_p \theta_0 \pi_e) i \right]^{-1}, \quad \varphi^\dagger \equiv c_p \theta_0 (\hat{\pi} - \pi_e)
\]

\[
dt = 0.5 \ dt_{\text{soundproof}}, \quad (\nabla z) \cdot \mathbf{v}^{n+1/2} \partial_z \pi_e \sim \frac{g}{\theta_e} \quad \Rightarrow \quad \text{a more robust Helmholtz problem}
\]
A robust Helmholtz problem for compressible NFT model

formulate the inhomogeneous AP for pressure perturbation (rather than for full pressure, to then derive the Helmholtz problem for pressure perturbation) in the spirit of the equation

\[
\frac{d\pi'}{dt} = -\gamma\pi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \pi_e \quad \Rightarrow \quad \frac{\partial \rho \pi'}{\partial t} + \nabla \cdot (\rho \pi' \mathbf{u}) = -\gamma \rho \pi \nabla \cdot \mathbf{u} - \rho \mathbf{u} \cdot \nabla \pi_e
\]

\[
\frac{\partial \rho^* \pi'}{\partial t} + \nabla \cdot (\rho^* \mathbf{v}' \pi') = - \left[ \gamma \rho^* \pi \frac{1}{G} \nabla \cdot (G \mathbf{v}) + \nabla \cdot (\rho^* \pi_e \mathbf{v}) - \pi_e \nabla \cdot (\rho^* \mathbf{v}) \right] - \beta (\varphi - \hat{\varphi})
\]

And how does one solve this “thing”?
A bespoke unstructured finite-volume discretisation in the horizontal

An unstructured median-dual mesh in 2D. The edge connecting the nodes $i$ and $j$ of the primary polygonal mesh intersects, at its mid-point, the face $S_j$ shared by computational dual cells surrounding the nodes $i$ and $j$; the white circles represent the centres of mass of the primary mesh, while the blue and black lines belong to the primary and the dual mesh, respectively.
A bespoke unstructured finite-volume discretisation in the horizontal enables simulation of global flows on the IFS reduced Gaussian grid.

Two primary meshes generated on N24 reduced Gaussian grid points with an approximate resolution of 3.75° (415 km). N24 indicates 24 latitudes between a pole and the equator. Gaussian grids are latitude-longitude grids. In reduced Gaussian grids, the number of grid points on latitudes near the poles is reduced to achieve a more even spacing of grid points. The shading represents the dual resolution, computed as the square root of the local dual volume. Such meshes support both control volume and spectral transform discretisations.
An equal-area MPI/OpenMP parallelisation of the IFS:

The computational mesh shown in the left panel of the preceding figure is here mapped onto tasks (32 partitions). Also shown is the internal halo of one partition, and the periodic halo responsible for the periodic boundary condition.

1600 partitions as typically applied in the new TCo1279 octahedral reduced Gaussian grid
Nonoscillatory forward-in-time semi-implicit integrators (for buoyant, rotational and acoustic modes) provide robust conservative solutions for inviscid multiscale problems.

Inviscid baroclinic instability (day 8, octahedral mesh N800): (top) contours of vertical velocity [cm/s] overlaid with isentropes (cnt. interval 5 °C) in the vertical cross section at 53°N, and (bottom) contours of surface meridional velocity [m/s] and the isentropes. The steepening of the isentropes together with the radiation of mesoscale gravity waves and the generation of grid-scale features are indicative of frontal collapse.
implicit LES (ILES) property of nonoscillatory forward-in-time numerics

Surface kinetic energy spectra at day 9 of the instability evolution, simulated with various horizontal resolutions on the octahedral meshes. For reference, the -3 and -5/3 slopes are shown with solid and dashed lines, respectively.
Generalised time-dependent curvilinear coordinate formulations enables a range of coordinate systems, especially in the vertical.

Trapped gravity wave (essentially nonhydrostatic) in the lee of a hill on a tiny planet. The hill can be seen in top panel, near the lower left corner. Red and blue patterns show updrafts and downdrafts, respectively, with contours of vertical velocity in m/s, in (top) vertical cross section along the equator, and (bottom) horizontal cross section at 3 km above the surface. Octahedral N128 mesh was used.
Subsequent advancements: real weather

FVM N128 simulation of a global circulation using realistic orography: \( w \) [m/s] at about 4 km above the surface reveals baroclinic eddies in the mid-latitudes and fine-scale features in the equatorial area and mountainous regions indicative of convection and gravity waves.
Subsequent advancements: real weather

FVM N640 simulation of a global circulation using realistic orography: surface pressure

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