Part 1: A geometric framework for parameterising ocean eddies

and

Part 2: A simple model of eddy saturation

Why is the Antarctic Circumpolar Current volume transport less sensitive to the wind stress in eddy-permitting models?

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Part 1: A geometric framework for parameterising ocean eddies

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1° (climate) resolution

$Re \sim 1$

1/12° resolution

$Re \sim 10^{10}$

(MICOM, University of Miami)
Ocean models behave very differently with parameterised and explicit eddies.

atmospheric box - CO₂ well mixed

ocean box - circulation, carbon cycle

(Munday et al., 2013)

“eddy saturation”

meridional section of neutral density: (Koltermann et al., 2011)

circumpolar volume transport ↔ global stratification

⇒ Southern Ocean eddies important for setting:
  - equilibrium global stratification;
  - equilibrium atmospheric CO₂;
  - ocean heat uptake;
  - ocean carbon uptake;
  - global sea level change;
  - ocean adjustment time scale.
Classical paradigm for location/structure of ocean eddies:

Eady (1949) model of baroclinic instability

- uniform rotation
- uniform stratification
- uniform shear
- opposing potential vorticity gradients at upper and lower boundaries

Most unstable mode:

Energy growth rate for most unstable mode:

\[ 0.61 \frac{f_0}{N_0} \frac{\partial u}{\partial z} \sim 0.3 \text{ day}^{-1} \text{ · atmosphere} \]
\[ 0.03 \text{ day}^{-1} \text{ · ocean} \]

Length scale of instability characterised by Rossby deformation radius:

\[ L_d = \frac{N_0 H}{f_0} \sim 1000 \text{ km} \text{ · atmosphere} \]
\[ 50 \text{ km} \text{ · ocean} \]
**Gent and McWilliams (1990):**

*adiabatic parameterisation of baroclinic instability*

Eddies mix along isopycnals (Redi 1982) ...

... and advect by an *eddy bolus velocity* - flattens isopycnals (Gent et al. 1995)

\[
\mathbf{u}^* = \frac{\partial}{\partial z} \left( \kappa \nabla b \right), \quad \mathbf{w}^* = -\nabla \cdot \left( \kappa \frac{\nabla b}{\partial b/\partial z} \right),
\]

removes available potential energy

Can relate eddy diffusivity, \( \kappa \), to mean flow (e.g., Visbeck et al. 1997)

or eddy energy (Eden and Greatbatch 2008)

**Alternative paradigm: potential vorticity mixing**

Often advocated ... rarely implemented in ocean GCMs!

Idea: potential vorticity \( q = \frac{f + \xi}{h} \) is materially conserved in absence of forcing/dissipation:

\[
\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0
\]

Stirred and mixed along isopycnals \( \Rightarrow \) down-gradient closure, \( \overline{q'u'} = -\kappa \nabla \rho \overline{q} \)?
PV mixing problem 1: conservation of energy
e.g., freely-decaying turbulence over a seamount (Adcock and Marshall, 2000)

Fig. 1. Schematic diagram illustrating a difficulty with eddy closures based on unconstrained potential vorticity mixing (adapted from Adcock and Marshall, 2000). Flow is confined to an abyssal layer, underlying an infinitely-deep, motionless upper layer. The initial state (left-hand panel) consists of a set of geostrophically-balanced eddies, associated with a deformed layer interface (solid line), above a seamount (solid shading). If the eddies were to completely homogenize the potential vorticity field (right-hand panel) this would require the layer interface to rise completely over the seamount and, in turn, a large anticyclonic circulation around the seamount. However, the energy of this hypothetical end state exceeds that in the initial state, indicating that unconstrained potential vorticity mixing is physically impossible.

PV mixing problem 2: conservation of momentum
periodic channel:

\[
\int \int q v' \, dx \, dy \, dz = 0
\]
note: this is the Charney-Stern stability condition

not satisfied by down-gradient potential vorticity closures without additional constraints (Green, 1970; J. Marshall, 1981)

e.g., here \( \overline{q'v'} = -\kappa \partial q / \partial y \) only consistent if \( \kappa = 0 \)

note: this is the Charney-Stern stability condition
Take-home message:

Eddies mix potential vorticity along density surfaces ...
... subject to constraints of energy and momentum conservation

Goal of this work:

Develop framework for interpreting and parameterising eddy potential vorticity fluxes in which the relevant symmetries and conservation laws are preserved.

Work with quasi-geostrophic “residual-mean” equations:

\[
\frac{\partial \Pi_g}{\partial t} + \ldots = -k \times \nabla \mathbf{u}' \\
\text{eddy force}
\]

(Maddison and Marshall, 2013) generalises to thickness-weighted average primitive equations (Young, 2012)

Key idea:

Write eddy potential vorticity flux (or eddy force) as divergence of an eddy stress tensor:

\[
\nabla \cdot \left( \begin{array}{ccc}
-N & M - P & 0 \\
M - P & N & 0 \\
R & S & 0 \\
\end{array} \right) = \nabla \cdot \mathbf{q}' \mathbf{u}'
\]

(Plumb 1986) “Taylor identity”

where:

\[
M = \frac{u'^2 - w'^2}{2} \quad N = -u'v' \quad \text{Reynolds stresses}
\]

\[
P = \frac{\nabla \cdot \mathbf{q}'}{2N_0^2} \quad \text{eddy potential energy}
\]

\[
R = \frac{f_0}{N_0^2} \frac{u'^2}{w'^2} \quad S = \frac{f_0}{N_0^2} \frac{w'^2}{u'^2} \quad \text{eddy buoyancy flux / “eddy form stress”}
\]
Why do this?!!!

\[ \overline{\mathbf{q}^T \mathbf{u}^T} = \nabla \cdot \begin{pmatrix} -N & M - P & 0 \\ M - P & N & 0 \\ R & S & 0 \end{pmatrix} \]

1. This is a mathematical identity! (down-gradient flux ≠ divergence of a tensor)

2. Momentum constraints preserved with appropriate boundary conditions:

\[ \frac{\partial \overline{\mathbf{u}^T g}}{\partial t} + \cdots = \nabla \cdot \text{(eddy momentum fluxes)} \]

3. Reduces to Gent and McWilliams (1990) / Greatbatch and Lamb (1990)
if we parameterise only the vertical momentum fluxes.

Therefore a natural framework for extending Gent and McWilliams.

Why do this?!!!

\[ \overline{\mathbf{q}^T \mathbf{u}^T} = \nabla \cdot \begin{pmatrix} -N & M - P & 0 \\ M - P & N & 0 \\ R & S & 0 \end{pmatrix} \]

4. Suppose we solve an **eddy energy** equation (Eden and Greatbatch, 2008):

\[ \frac{\partial E}{\partial t} + \cdots = -\overline{\mathbf{u}^T g} \cdot \nabla \text{(eddy force)} = \overline{\mathbf{u}^T g} \cdot \mathbf{k} \times \overline{\mathbf{q}^T \mathbf{u}^T} \]

This **eddy energy** gives a bound on the **magnitude of the eddy stress tensor**:

\[ \frac{1}{2} \left[ (-N)^2 + (M - P)^2 + (M + P)^2 + N^2 + \frac{N^2}{f_0^2} (R^2 + S^2) \right] \leq E^2 \]

This means there are **no remaining dimensional unknowns**!
Why do this?!!!

5. This allows us to rewrite the eddy stress tensor in terms of the eddy energy, two non-dimensional eddy anisotropies, and three eddy angles:

\[ M = -\gamma_m E \cos 2\phi_m \cos^2 \lambda \quad N = \gamma_m E \sin 2\phi_m \cos^2 \lambda \quad P = E \sin^2 \lambda \]

\[ R = \gamma_b \frac{f_0}{N_0} E \cos \phi_b \sin 2\lambda \quad S = \gamma_b \frac{f_0}{N_0} E \sin \phi_b \sin 2\lambda \]

e.g., barotropic eddies: \( \gamma_m = 0 \) (plan view) \( \gamma_m \to 1 \) "wave-like"

Why do this?!!!

6. Eddy angles have a strong connection with classical stability theory:

eddies lean “against” mean shear \( \Rightarrow \) extract energy from mean flow - instability;
eddies lean “into” mean shear \( \Rightarrow \) return energy to mean flow - stability.

(Waterman et al. 2011)
Application: Eady model

Eady energy budget:

\[
\frac{\partial}{\partial t} \iiint E \, dx \, dy \, dz = - \iiint \bar{u} \bar{q} \bar{v} \, dx \, dy \, dz
\]

\[
= - \iiint \bar{u} \frac{\partial \bar{S}}{\partial z} \, dx \, dy \, dz
\]

substitute eddy stress tensor

\[
= \iiint \frac{\partial \bar{\Pi}}{\partial z} S \, dx \, dy \, dz
\]

integrate by parts

\[
= \alpha \frac{f_0}{N_0} \frac{\partial \bar{\Pi}}{\partial z} \iiint E \, dx \, dy \, dz
\]

apply energy bound

\[-1 \leq \alpha \leq 1 \quad \text{Eady growth rate}
\]

if \( \alpha = 0.61 \)

can interpret as an eddy diffusivity

- turns out to be Visbeck et al. (1997)

How anisotropic are the eddies?

3-layer QG model

(a) \( H_1 \psi_1 \) (Sv)
(b) \( H_2 \psi_2 \) (Sv)
(c) \( H_3 \psi_3 \) (Sv)
(d) \( q_1 \) (10\(^{-4}\) s\(^{-1}\))
(e) \( q_2 \) (10\(^{-4}\) s\(^{-1}\))
(f) \( q_3 \) (10\(^{-4}\) s\(^{-1}\))
eddy anisotropies

layer 1                         layer 2                        layer 3

buoyancy

momentum

\[ N = \gamma_m E \sin 2\phi_m \cos^2 \lambda \]
\[ S = \gamma_b \frac{f_0}{N_0} E \sin \phi_b \sin 2\lambda \]

What sets the eddy angles?   - for linear Rossby waves: refraction

(e.g., Buhler and McIntyre, 2005)
Ray-tracing - barotropic jet  \( (\text{Tamarin et al.}, 2015) \)

Up-gradient momentum transfer in ACC  \( (\text{Klocker et al.}, 2015) \)
Mixing of potential vorticity?

If we: (i) solve an eddy potential enstrophy \( \overline{q^2} \) budget;
(ii) include dissipation of \( \overline{q^2} \) (= potential vorticity mixing);
(iii) ensure \( \overline{q'u'} \) vanishes when \( \overline{q^2} \) vanishes;

[use another bound on divergence of eddy stress tensor?]

then **Arnold’s first stability theorem is preserved.**

**Physical interpretation?**  (Marshall and Adcroft, 2010)

Eddy energy equation:

\[
\frac{\partial}{\partial t} \frac{\overline{u' \cdot u'}}{2} + \nabla \cdot (\ldots) = + \overline{q'u' \cdot \nabla \overline{\psi}}
\]

Eddy enstrophy equation:

\[
\frac{\partial}{\partial t} \frac{\overline{q'^2}}{2} + \nabla \cdot (\ldots) = - \overline{q'u' \cdot \nabla q}
\]

If \( dq/d\overline{\psi} > 0 \), eddy energy can grow only at the expense of eddy potential enstrophy.

\[ \Rightarrow \text{stable (in the sense of Lyapunov) - **Arnold’s first stability theorem.**} \]

**Coordinate-invariant derivation**  (Maddison and Marshall, 2013)

quasigeostrophic PV equation:

\[
\partial_t \overline{q} + \left[ \frac{[u_g]^\alpha}{q} \right]_{,\alpha} = - T_{;ab}^{ab}
\]

double divergence

\[ \Rightarrow 2 \text{ forms of gauge freedom} \]

eddy flux tensor

\[
g_{ac}T^{cb} = \begin{pmatrix}
N & M - K & R \\
M + K & -N & S \\
0 & 0 & 0
\end{pmatrix}
\]

“residual-mean”

\[
\begin{pmatrix}
N & M + K & 0 \\
M - K & -N & 0 \\
R & S & 0
\end{pmatrix}
\]

Cronin (1996)

\[
\begin{pmatrix}
N & M - K & 0 \\
M + K & -N & 0 \\
R & S & 0
\end{pmatrix}
\]

“half-residual mean”

\[
\begin{pmatrix}
N & M & \frac{1}{2}R \\
M & -N & \frac{1}{2}S \\
\frac{1}{2}R & \frac{1}{2}S & 0
\end{pmatrix}
\]

Plumb (1986)

\[
\begin{pmatrix}
N & M - P & 0 \\
M + P & -N & 0 \\
R & S & 0
\end{pmatrix}
\]

Hoskins et al. (1983) “E-vector”

\[
\begin{pmatrix}
N & 2M & 0 \\
0 & -N & 0 \\
R & S & 0
\end{pmatrix}
\]

Approach generalises to isopycnal thickness-weighted primitive equations

(cf. Young, 2012)
Part 2: A simple model of eddy saturation

Q: Why is the Antarctic Circumpolar Current volume transport less sensitive to the wind stress in eddy-permitting models?

A nonlinear oscillator describing storm track variability†

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baroclinity: \( \dot{s} = F - f \)
heat flux: \( \dot{f} = 2(s - s_0)f \)

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†For an online version of this paper, see: https://doi.org/10.1002/qj.3652
Eddy-resolving box model of the ACC
(Munday et al., 2013)

integrate to equilibrium (1000s of years) with explicit eddies:

atmospheric box - CO₂ well mixed

ocean box - circulation, carbon cycle

(cf. Wolfe and Cessi, 2010; ... ; ... )

Circumpolar volume transport - eddy saturation
(Munday et al., 2013)

Simple model of eddy saturation

**momentum budget**  
(Johnson and Bryden, 1989)

\[
\text{wind stress} \approx \text{eddy form stress}
\]

\[
\bar{v}^2 \leq \sqrt{EKE \times EPE}
\]

\[
\bar{v}^2 = \alpha \cdot \text{eddy energy}
\]

(Marshall et al., 2012)

**eddy energy budget**

\[
\text{eddy energy source} \approx \text{eddy energy sink}
\]

\[
\text{shear} \times \text{eddy energy} = \text{dissipation rate} \times \text{eddy energy}
\]

\[
O(\text{months})
\]

(Zhai and Marshall, 2015)

**eddy energy ~ wind stress**

**volume transport ~ dissipation rate**

**Numerical model**

MITgcm, 10km resolution, Munday et al. (2015) configuration with a ridge
Eddy kinetic energy

Volume transport
Q: suppose we increase the bottom drag $\Rightarrow$ increased eddy energy dissipation
- what balances this?

A: increased wind work $\Rightarrow$ surface currents must strengthen

wind work in SOSE:

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baroclinity: $\dot{s} = F - f$
heat flux: $\dot{f} = 2(s - s_0)f$
**2-d dynamical system for ACC** (cf. Ambaum and Novak, 2015 for atmospheric storm track)

\[ \frac{dT}{dt} = \frac{L}{4\Lambda^2} \left( \frac{\tau_s}{\rho_0} - \alpha \frac{f_0}{N_0} E_{\text{eddy}} \right), \]
\[ \frac{dE_{\text{eddy}}}{dt} = \left( \frac{2\alpha f_0}{LH^2 N_0} T - r \right) E_{\text{eddy}} + r E_0. \]

Extra term to maintain minimum eddy energy, \( E_0 \)

Spin-up time scale ~ 1000 years (cf. Allison et al., 2011)

**Implications for eddy parameterisation**

Can we capture eddy saturation with parameterised eddies through a simple extension of Gent and McWilliams (1990)?

Gent and McWilliams can be interpreted as parameterising the eddy form stress:

\[ \overline{v'z'} = -k \nabla \overline{z} \]

Here: \( \overline{v'z'} = \alpha \cdot \text{eddy energy} \)

(Marshall et al., 2012)

Is an essential ingredient to obtain eddy saturation.

Idea: retain Gent and McWilliams (1990);

- solve a parameterised eddy energy equation (following Eden and Greathbatch, 1998);
- rescale the eddy diffusivity profile to match the eddy energy in the vertical integral.
Summary of key points

- Geostrophic eddies are fundamental in setting the structure and circulation of the ocean.

- Preserving symmetries and conservation laws in models with parameterised eddies ⇒ classical stability criteria carry over.

- New geometric framework for diagnosing and interpreting eddy-mean flow interactions.

- Simple model of eddy saturation:
  - eddy energy scales with wind stress;
  - ACC volume transport scales with bottom dissipation - implications for climate models?

- Much left to do:
  - implement and validate extension to Gent and McWilliams to capture eddy saturation;
  - simple extension to Gent and McWilliams to sharpen jets;
  - eddy-topography interactions;
  - diagnostics of eddy-mean flow interactions;
  - applications to planetary atmospheres?